Numerical Solution of Initial Boundary Value Problems

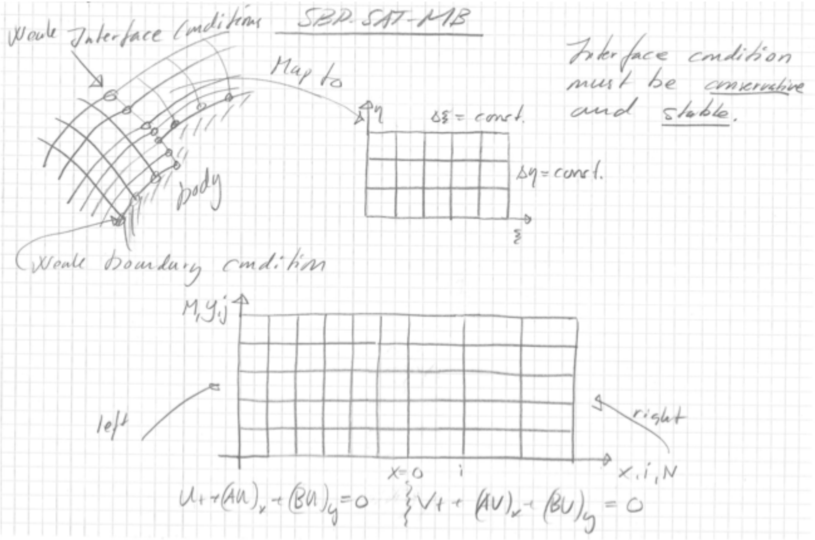
Jan Nordström
Division of Computational Mathematics
Department of Mathematics

Linköping University

Multiple dimensions: FDM, FVM, DG

- All the methods discussed below are of SBP-SAT type.
- Stability and conservation should be guaranteed.
- Overlapping/sliding methods not included.

FDM on SBP-SAT form



Kronecker products

For arbitrary matrices $A \in^{m\times n}$ and $B \in^{p\times q}$, the Kronecker product is defined as

$$A \otimes B = \begin{bmatrix} a_{1,1}B & \dots & a_{1,m}B \\ \vdots & \ddots & \vdots \\ a_{n,1}B & \dots & a_{m,n}B \end{bmatrix}. \tag{1}$$

The Kronecker product is bilinear, associative and obeys

$$(A \otimes B)(C \otimes D) = (AC \otimes BD) \tag{2}$$

if the usual matrix products are defined. For inversion and transposing we have

$$(A \otimes B)^{-1,T} = A^{-1,T} \otimes B^{-1,T} \tag{3}$$

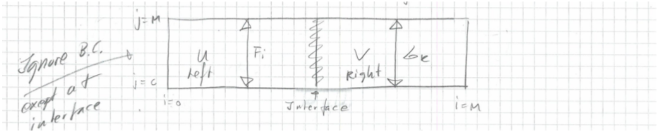
if the usual matrix inverse is defined.

Organisation

$$\frac{\partial F}{\partial x} \approx (P_x^{-1}Q_x \otimes I_y)\vec{F} = (P_x^{-1}Q_x \otimes I_y \otimes I_4)\vec{F} = (P_x^{-1}Q_x \otimes I_y \otimes A)\vec{u}$$

$$\frac{\partial G}{\partial y} \approx (I_x \otimes P_y^{-1}Q_y)\vec{G} = (I_x \otimes P_y^{-1}Q_y \otimes I_4)\vec{G} = (I_x \otimes P_y^{-1}Q_y \otimes B)\vec{u}$$

$$\vec{F} = \begin{bmatrix} F_0 \\ \vdots \\ F_i \\ \vdots \\ F_N \end{bmatrix} \rightarrow \begin{bmatrix} F_{i0} \\ \vdots \\ F_{ij} \\ \vdots \\ F_{iM} \end{bmatrix} \rightarrow \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}_{ij} \rightarrow A \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$



The semi-discrete schemes

Focus only at the interface:

$$u_t + (P_x^{-1} Q_x \otimes I_y)\vec{F} + (I_x \otimes P_y^{-1} Q_y)\vec{G} = (P_x^{-1} \otimes I_y)\Sigma_L (u - v)$$

$$v_t + (P_x^{-1} Q_x \otimes I_y)\vec{F} + (I_x \otimes P_y^{-1} Q_y)\vec{G} = (P_x^{-1} \otimes I_y)\Sigma_R (v - u)$$

Determine Σ_L, Σ_R such that the coupling is stable and conservative.

Conservation $\quad \phi = \phi(x,y,t) = \text{smooth}, \phi(\pm\infty,\pm\infty,t) = 0.$

$$\phi^T(P_x \otimes P_y)u_t + \phi^T(Q_x \otimes P_y)F + \phi^T(P_x \otimes Q_y)G = \phi^T(I_x \otimes P_y)\Sigma_L(u-v)$$

$$\begin{aligned} \text{SBP:}\ \ Q_x &\rightarrow -Q_x^T + B_x \\ Q_y &\rightarrow -Q_y^T + B_y \end{aligned}$$

$$\begin{aligned} &\phi^T(P_x \otimes P_y)u_t - \phi^T(Q_x^T \otimes P_y)F - \phi^T(P_x \otimes Q_y^T)G + \\ &\phi^T(B_x \otimes P_y)F + \phi^T(P_x \otimes B_y)G = \phi^T(I_x \otimes P_y)\Sigma_L(u-v) \end{aligned}$$

Note $\phi^T(Q_x^T \otimes P_y)F = \phi^T(Q_x^T \otimes I_y)(I_x \otimes P_y)F =$

$$\underbrace{\phi^T(Q_x^T \otimes I_y)((P_x^{-1})^T \otimes I_y)}_{\phi_x^T}\underbrace{(P_x \otimes I_y)(I_x \otimes P_y)}_{(P_x \otimes P_y)}F = \phi_x^T(P_x \otimes P_y)F$$

Same procedure as in one dimension, all derivatives flipped $\Rightarrow$

$$\phi^T(P_x \otimes P_y)u_t - \phi_x^T(P_x \otimes P_y)F - \phi_y^T(P_x \otimes P_y)G =$$

Boundary terms at $i = 0$ for all gridpoint $j = 0, M$.

$$\underbrace{-\phi_N^T P_y F_N + \phi^T(I_x \otimes P_y \otimes I_4)\underbrace{(E_N \otimes I_y \otimes \tilde{\Sigma_L})}_{\Sigma_L}(u - v)}_{-\phi_N^T(P_y^L \otimes A)u_N + \phi_N^T(P_y^L \otimes \tilde{\Sigma_L})(u_N - v_0) \;=\; IT_L}$$

From the other equation we get the corresponding terms

$$IT_R = \phi_0^T(P_y^R \otimes A)v_0 + \phi_0^T(P_y^R \otimes \tilde{\Sigma_R})(v_0 - u_N).$$

Conservation at interface requires $IT_L + IT_R = 0$, hence

$$IT_L + IT_R = (\phi_0 = \phi_N = \phi_i,\ \ P_y^L = P_y^R = P_y) =$$

$$\phi_i^T (P_y \otimes (-A + \tilde{\Sigma_L} - \tilde{\Sigma_R})(u_N - v_0)) = 0$$

lead to conservation.

$$\therefore\ \ \tilde{\Sigma_L} = \tilde{\Sigma_R} + A \text{ is the conservation condition.}$$

Note similarity with 1D $\sigma_L = \sigma_R + a$

The same technique, multiplying from left with $u^T(P_x \otimes P_y) \Rightarrow$

$$\frac{d}{dt}(u^T(P_x^L \otimes P_y^L)u + v^T(P_x^R \otimes P_y^R)v) =$$

$$\begin{bmatrix} u_N \\ v_0 \end{bmatrix}^T \underbrace{\begin{bmatrix} P_y \otimes (-A + 2\Sigma_L) & P_y \otimes (-\Sigma_L - \Sigma_R) \\ P_y \otimes (-\Sigma_R - \Sigma_L) & P_y \otimes (A + 2\Sigma_R) \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} u_N \\ v_0 \end{bmatrix}$$

Recall $A \otimes B = a_{ij}B$, Unfortunately, this is not the form of $\tilde{A}$.

However there exists ψ such that $P_y \otimes A_{ij} = \psi^T(A_{ij} \otimes P_y)\psi$.
"even permutation similar. " $\Rightarrow$

$$\begin{bmatrix} u_N \\ v_0 \end{bmatrix}^T \begin{bmatrix} \psi^T(A_{11} \otimes P_y)\psi & \psi^T(A_{12} \otimes P_y)\psi \\ \psi^T(A_{12} \otimes P_y)\psi & \psi^T(A_{22} \otimes P_y)\psi \end{bmatrix} \begin{bmatrix} u_N \\ v_0 \end{bmatrix}$$

$$\begin{bmatrix} \psi u_N \\ \psi v_0 \end{bmatrix}^T \underbrace{\begin{bmatrix} A_{11} \otimes P_y & A_{12} \otimes P_y \\ A_{12} \otimes P_y & A_{22} \otimes P_y \end{bmatrix}}_{\underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix}}_{\tilde{\tilde{A}}} \otimes P_y} \begin{bmatrix} \psi u_N \\ \psi v_0 \end{bmatrix}$$

For stability we need $\tilde{\tilde{A}} \leq 0$. The conservation condition in $\tilde{\tilde{A}} \Rightarrow$

$$\begin{bmatrix} -A + 2\Sigma_L & -2\Sigma_L + A \\ -2\Sigma_L + A & -A + 2\Sigma_L \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{\lambda=0,2} \otimes \underbrace{\begin{bmatrix} -A + 2\Sigma_L \end{bmatrix}}_{\leq 0}.$$

$\therefore$ Stability if conservation condition holds and $-A + 2\Sigma_L \leq 0$.

We need $2\Sigma_L - A \leq 0$. However $A = X\Lambda X^T = X(\Lambda^+ + \Lambda^-)X^T$.

Let $\Sigma_L = X\tilde{\Sigma_L}X^T \Rightarrow$

$$2\Sigma_L - A = X^T(2\bar{\Sigma}_L - \Lambda^+ - \Lambda^-)X = -X\Lambda^+X^T + X(2\tilde{\Sigma}_L - \Lambda^-)X^T$$

1st choice $\Sigma_L = \frac{\Lambda^-}{2}$ damping by $-A^+$

2nd choice $\Sigma_L = \Lambda^-$ damping by $A^- - A^+$

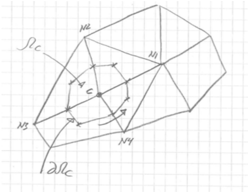
Summary of the SBP-SAT-MB technique

- It is a conservative and stable method.
- Nodes on interface must coincide.
- The integration operators (norms) must be the same on both sides of interface.
- If the norms are different, interpolation operators must be used.
- If the nodes do not coincide, interpolation operators must be used.
- The stability and conservation conditions are similar to the one-dimensional ones.

Node centered unstructured finite volume methods

SBP property

Consider



$$u_t + u_x = 0 \tag{4}$$

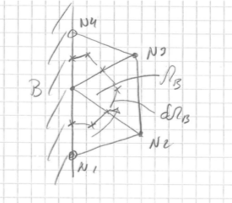
$$\underbrace{\int_{\Omega_c} u_t dxdy}_{\Omega_c u_{ct}} + \underbrace{\int_{\Omega_c} u_x dxdy}_{\text{Green's formula} \Rightarrow \oint_{\partial\Omega_c} u dy} = 0 \tag{5}$$

$$\oint_{\partial\Omega_c} u dy \approx \sum_i \frac{u_c + u_i}{2}\Delta y_i = \underbrace{\sum_i \frac{u_c}{2}\Delta y_i}_{\text{closed loop=0}} + \underbrace{\sum_i \frac{u_i \Delta y_i}{2}}_{\text{Matrix elements}}$$

$Q_{cNi} = \dfrac{\Delta y_i}{2}$ counter clockwise, $Q_{Nic} = -\dfrac{\Delta y}{2} \Rightarrow Q$ skew-symmetric

SBP operators

Boundaries



$$\oint_{\partial\Omega_B} u dy \approx \sum_i \frac{u_B + u_{Ni}}{2}\Delta y_i + u_B \Delta y_B$$

$$= u_B\underbrace{\left(\sum_i \frac{\Delta y_i}{2} + \Delta y_B\right)}_{\frac{-\Delta y_B}{2} + \Delta y_B} + \sum_i \frac{u_{Ni}\Delta y_i}{2} =$$

$$\therefore \quad Q_{BB} = \frac{\Delta y_B}{2}, \qquad Q_{BNi} = \frac{\Delta y_i}{2} = -Q_{NiB}$$

$\therefore \quad Q_x + Q_x^T = \Delta Y,$ elements of $\Delta Y \neq 0$ only at boundary points.

$\therefore \quad Q_y^T + Q_y^T = -\Delta X,$ elements of $\Delta X \neq 0$ only at boundaries.

$\therefore$ The UFVM yields SBP operators.

Boundary procedures

What about boundary bonditions and SAT?

$$\begin{aligned} u_t + Au_x + Bu_y &= 0, \ \vec{x} \in \Omega \\ Lu &= g, \ \vec{x} \in \partial\Omega \\ u(\vec{x}, 0) &= f, \ \vec{x} \in \Omega \end{aligned}$$

Let A, B be symmetric and consider the energy.

$$\int_\Omega uu_t dxdy + \int_\Omega u^T (Au)_x + u^T (Bu)_y dxdy = 0 \ \Rightarrow$$

$$\frac{1}{2} \|u\|_t^2 + \frac{1}{2} \int_\Omega (u^T Au)_x + (u^T Bu)_y dxdy = 0 \Rightarrow$$

Boundary procedures

$$\|u\|_t^2 + \oint_{\partial\Omega} u^T A u dy - u^T B u dx = 0 \Rightarrow$$

$$\|u\|_t^2 + \oint_{\partial\Omega} u^T \underbrace{((A, B) \cdot \vec{n})}_{\tilde{A}=symmetric} u ds = 0, \quad \tilde{A} = X \Lambda_A X^T, \quad \Lambda_A = \Lambda_A^+ + \Lambda_A^- \Rightarrow$$

$$\|u\|_t^2 + \oint_{\partial\Omega} u^T X(\Lambda_A^+ + \Lambda_A^-) X^T u \, ds = 0, \quad \Lambda_A^- \text{ dangerous.}$$

Use the characteristic boundary condition $(X^T u)^- = (X^T \bar{u})^- \Rightarrow$

$$\frac{d}{dt}\|u\|^2 + \oint_{\partial\Omega} u^T A^+ u + \bar{u}^T A^- \bar{u} ds = 0. \qquad (6)$$

$\therefore$ A well-posed problem.

Conservation

Is the scheme $Pu_t + (Q_x \otimes A)u + (Q_y \otimes B)u = 0$ conservative ?

$$\phi^T P u_t + \phi^T (Q_x \otimes A)u + \phi^T (Q_y \otimes B)u = 0$$

$$Q_x = -Q_x^T + \Delta Y, \quad Q_y = -Q_y^T - \Delta X \Rightarrow$$

$$\phi^T P u_t - \phi^T (Q_x^T \otimes A)u - \phi^T (Q_y^T \otimes B)$$

$$+\phi^T (\Delta Y \otimes A)u + \phi^T (-\Delta X \otimes B)u = 0$$

$$\phi^T P u_t - \left[(Q_x \otimes I)\phi\right]^T [I \otimes A]\, u - \left[(Q_y \otimes I)\phi\right]^T [I \otimes B]\, u$$

$$+ \underbrace{\sum_i \phi_i((A,B) \cdot \vec{n_i}) u_i ds_i}_{BT}$$

$\therefore$ No remaining interiour terms: Conservative!

Stability of UFVM

$$(P \otimes I)u_t + (Q_x \otimes A)u + (Q_y \otimes B)u = (E_B^T \otimes I)\underbrace{\Sigma(u - G)}_{H}$$

E_B picks out boundary points. Non-zero on the boundary only.

$$u^T(P \otimes I)u_t + u^T(Q_x \otimes A) + u^T(Q_y \otimes B)u = u^T(E_B^T \otimes I)H =$$

$$= ((E_B \otimes I)u)^T H = u_B^T H = \sum_{i \in B} u_{Bi}^T H_i = \sum u_i^T \Sigma_i (u - \bar{u})_i$$

$$(\|u\|^2_{P \otimes I)})_t + u^T(\Delta Y \otimes A)u + u^T(-\Delta X \otimes B)u = 2u_B^T H.$$

$$\begin{aligned}
(\|u\|^2_{P \otimes I})_t &= \sum -u_i^T((A, B) \cdot \vec{n}_i)u_i ds_i + 2u_i^T \Sigma_i (u_i - \bar{u}_i) \\
&= \sum_i \left(-u_i^T \tilde{A}_i u_i + 2u_i^T \tilde{\Sigma}_i (u_i - \bar{u}_i)\right) ds_i
\end{aligned}$$

Energy estimate

$$\tilde{A}_i = \tilde{A}_i^+ + \tilde{A}_i^- \Rightarrow \sum_i \left(-u_i^T A_i^+ u_i - u_i^T (A_i^- - 2\tilde{\Sigma}_i) u_i - 2u_i \tilde{\Sigma}_i \bar{u}_i\right) ds_i.$$

Let $\Sigma_i = +A_i^- \Rightarrow$

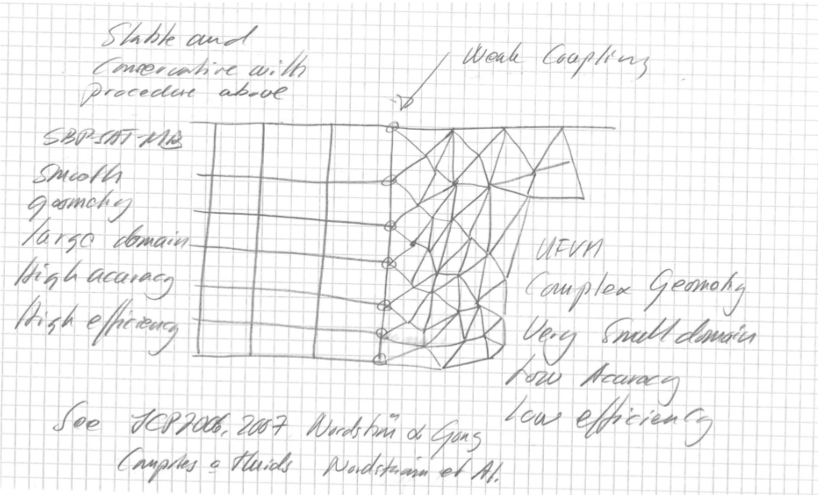
$$\frac{d}{dt}(\|u\|^2_{P\otimes I}) + \sum_{i\in B} u_i^T A^+ u_i + \bar{u}_i^T A^- \bar{u}_i ds_i = \underbrace{\sum_{i\in B} (u - \bar{u}_i)^T A_i^- (u - \bar{u}_i) ds_i}_{\leq 0}.$$

$\therefore$ An estimate in terms of data !

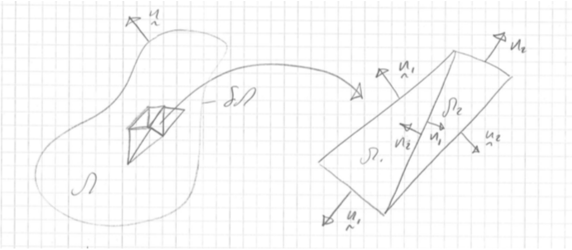
$\therefore$ UFVM both conservative and stable.

$\therefore$ Warning ! Could be suprisingly inaccurate with O(1) errors.

SBP-SAT-MB + UFVM = Hybrid



Discontinuous Galerkin (DG)



$$u_t + (Au)_x + (Bu)_y = 0, \;\; F = Au, \;\; G = Bu, \;\; u = \text{solution vector}$$

$$A = A^T, \;\; B = B^T = \text{constant matrices}$$

1. Multiply with smooth function $\alpha = \alpha(x, y, t)$.

$$\int_{\Omega_i} \alpha u_t \, dx \, dy + \int_{\Omega_i} \alpha F_x + \alpha G_y \, dx \, dy = 0$$

Derivation of penalty term

2) Integrate-by-parts

$$\int_{\Omega_i} \alpha u_t \, dx \, dy + \int_{\Omega_i} (\alpha F)_x + (\alpha G)_y \, dx \, dy - \int_{\Omega_i} \nabla \alpha \cdot \vec{F} \, dx \, dy = 0$$

where $\vec{F} = (F, G)$. Green-Gauss $\Rightarrow$

$$\int_{\Omega_i} \alpha u_t d\Omega - \int_{\Omega_i} \nabla \alpha \cdot \vec{F} d\Omega + \oint_{\partial \Omega_i} \alpha F dy - \alpha G dx = 0$$

$$\int_{\Omega_i} \alpha u_t d\Omega - \int_{\Omega_i} \nabla \alpha \cdot \vec{F} d\Omega + \oint_{\partial \Omega_i} \alpha \vec{F} \cdot \vec{n} ds = 0$$

3) Change $F \rightarrow \hat{F}$ = numerical Flux= what you want it do be.

Derivation of penalty term

$$\int_{\Omega_i} \alpha u_t - \nabla\alpha \cdot F d\Omega + \oint_{\partial\Omega_i} \alpha\hat{F} \cdot \vec{n} ds = 0 \tag{7}$$

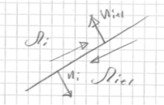
Integrate back $\Rightarrow$

$$\int_{\Omega_i} \alpha u_t + \alpha\nabla \cdot F d\Omega + \underbrace{\oint_{\partial\Omega_i} \alpha(\hat{F} - F) \cdot \vec{u} ds}_{\text{penalty term}} = 0 \tag{8}$$

A penalty term just as in the SBP-SAT technique.

Conservation

Look at two nearby elements. Let $\alpha = \phi$.



$$\int_{\partial\Omega_i} \phi_i \hat{F}_i \cdot \vec{n}_i ds + \int_{\partial\Omega_{i+1}} \phi_{i+1} \hat{F}_{i+1} \cdot \vec{n}_{i+1} ds =$$

$$= (\vec{n}_{i+1} = -\vec{n}_i, \phi_{i+1} = \phi_i) = \int_{\partial\Omega_i} \phi_i \hat{F}_i \cdot n_i + \phi_{i+1} \hat{F}_{i+1}(-\vec{n}_i) ds$$

$$= \int_{\partial\Omega_i} \phi_i (\vec{\hat{F}}_i - \vec{\hat{F}}_{i+1}) \cdot \vec{n}_i ds.$$

$\therefore$ No terms at the interface if $(\vec{\hat{F}}_i - \vec{\hat{F}}_{i+1}) \cdot \vec{n}_i = 0$.

$\therefore$ DG is a conservative if numerical fluxes on adjacent elements are the same.

Stability

Let $\alpha = u \Rightarrow$

$$\int_{\Omega_i} u u_t d\Omega + \int_{\Omega_i} u^T (Au)_x + u^T (Bu)_y d\Omega$$

$$+ \int_{\partial\Omega_i} u^T (\hat{F} - F) \cdot \vec{n} ds = 0$$

$$\frac{1}{2} ||u||_t^2 + \frac{1}{2} \int_{\partial\Omega_i} \underbrace{u^T Audy - u^T Bu\, dx}_{u^T F \cdot \vec{n} ds} + \oint_{\partial\Omega_i} u^T (\hat{F} - F) \cdot \vec{n} ds = 0$$

$$||u||_t^2 + 2 \oint_{\partial\Omega_i} u^T (\hat{F} - \frac{1}{2} F) \cdot \vec{n} ds = 0.$$

Stability

The other side $\Rightarrow$

$$(\|u\|_i^2 + \|u\|_{i+1}^2)_t + 2\oint_{\partial\Omega_i} u_i^T(\hat{F}_i^T - \frac{1}{2}F_i)\cdot\vec{n}_i ds + 2\oint_{\partial\Omega_{i+1}} u_{i+1}^T(\hat{F}_{i+1}^T - \frac{1}{2}F_{i+1})\cdot\vec{n}_{i+1} ds = 0$$

$$\hat{F}_i\cdot\vec{n}_i = C_0\{F\}_i + C_1[F]_i\,;\;\; \{F\}_i = (A,B)\cdot\vec{n}_i\left(\frac{u_i + u_{i+1}}{2}\right)$$

$$\hat{F}_{i+1}\cdot\vec{n}_i = C_2\{F\}_i + C_3[F]_i\,;\;\; [F]_i = (A,B)\cdot\vec{n}_i(u_i - u_{i+1})$$

Conservation $\Rightarrow (C_0 - C_2)\{F\}_i + (C_1 - C_3)[F]_i = 0$

$\Rightarrow \underline{C_2 = C_0, C_3 = C_1}$

Stability

$$
\begin{aligned}
BT = & \oint u_i^T(C_0\tilde{A}\left(\frac{u_i+u_{i+1}}{2}\right) + C_1\tilde{A}(u_i - u_{i+1}) - \frac{1}{2}\tilde{A}u_i) \\
& - u_{i+1}^T\left(C_0\tilde{A}\frac{u_i+u_{i+1}}{2} + C_1\tilde{A}(u_i - u_{i+1}) - \frac{1}{2}\tilde{A}u_{i+1}\right)ds = \\
= & \oint u_i^T(\frac{1}{2}(C_0 - I) + C_1)\tilde{A}u_i + u_i^T(\frac{C_0}{2} - C_1)u_{i+1} \\
& - u_{i+1}^T(\frac{1}{2}(C_0 - I) - C_1)\tilde{A}u_i - u_{i+1}^T(\frac{C_0}{2} - C_1)u_i ds
\end{aligned}
$$

Put on matrixform, to be able to see what is going on.

$$
\begin{aligned}
BT = & \oint \begin{bmatrix} u_i \\ u_{i+1} \end{bmatrix}^T \begin{bmatrix} (\frac{1}{2}(C_0 - I) + C_1)\tilde{A} & (\frac{C_0}{2} - C_1)\tilde{A} \\ -(\frac{C_0}{2} + C_1)\tilde{A} & (-\frac{1}{2}(C_0 - I) + C_1)\tilde{A} \end{bmatrix} \begin{bmatrix} u_i \\ u_{i+1} \end{bmatrix} ds = \\
= & \oint \begin{bmatrix} u_i \\ u_{i+1} \end{bmatrix}^T \begin{bmatrix} (C_1 + \frac{C_0 - I}{2})\tilde{A} & -C_1\tilde{A} \\ -C_1\tilde{A} & (C_1 - \frac{C_0 - I)}{2})\tilde{A} \end{bmatrix} \begin{bmatrix} u_i \\ u_{i+1} \end{bmatrix} ds.
\end{aligned}
$$

Stability

$$\text{Choose } C_0 = I \Rightarrow BT = \oint \begin{bmatrix} u_i \\ u_{i+1} \end{bmatrix}^T \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{\lambda=0,+1} \otimes C_1 \tilde{A} \begin{bmatrix} u_i \\ u_{i+1} \end{bmatrix} ds$$

Need to choose C_1 such that $C_1\tilde{A} \geq 0$.

$$\tilde{A} = X\Lambda X^T \quad C_1 = X\Sigma X^T \Rightarrow C_1\tilde{A} = X\Sigma\Lambda X^T = X^T\Sigma(\Lambda^+ + \Lambda^-)X$$

$$\text{Choose } \Sigma \text{ such that:} \quad \Sigma\Lambda^+ = \Lambda^+, \quad \Sigma\Lambda^- = \delta|\Lambda^-| \Rightarrow$$

$$C\tilde{A} = X(\Lambda^+ + \delta|\Lambda^-|)X^T = \begin{cases} A^+, \delta = 0 \\ |A|, \delta = 1 \end{cases}$$

Stability

With the above choices we find

$$\therefore\ BT = \oint \begin{bmatrix} u_i \\ u_{i+1} \end{bmatrix}^T \begin{bmatrix} |A| & -|A| \\ -|A| & |A| \end{bmatrix} \begin{bmatrix} u_i \\ u_{i+1} \end{bmatrix} ds = \oint (u_i - u_{i+1})^T |A| (u_i - u_{i+1}) ds \Rightarrow$$

and hence

$$\frac{d}{dt}(\|u\|_i^2 + \|u\|_{i+1}^2) = -\oint \underbrace{[u]\,|A|\,[u]}_{\text{jumps}} ds$$

$\therefore$ A stable scheme.

$\therefore$ Also expensive, interfaces everywhere, needed for high accuray.

Summary of multi-dimensional schemes

- SBP-SAT-MB: Highly efficient. Requires non-nasty geometry to be optimal.
- UFVM: Can handle complex geometry, low accuracy, slow.
- DG: Very stable and accurate but expensive for high order and multiple dimensions.
- Hybrid: Combines SBP-SAT-MB + UFVM or dG. Maybe optimal by combining the best properties.
- References

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End of Lecture 3