On the dynamic of the Euler-Korteweg system

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1 Introduction

2 Main results

3 Ideas for the second theorem

Euler-Korteweg system:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t u + u \cdot \nabla u + \nabla g(\rho) = \nabla \Big(K(\rho)\Delta\rho + \frac{1}{2} K'(\rho)|\nabla \rho|^2 \Big), \end{cases} \quad (x,t) \in \mathbb{R}^d \times [0,T]. \tag{EK}$$

$K(\rho)$: capillary coefficient, smooth, positive on $\mathbb{R}^{+*}$. K can be unbounded near $\rho = 0$, an important case is $K = 1/\rho$ which corresponds to the so called quantum pressure.

$g(\rho)$: related to the pressure by $g' = p'/\rho$, stability assumption: $g'(\rho_0) > 0$.

Gross-Pitaevskii equation

$$i\partial_t \psi + \Delta \psi = (|\psi|^2 - 1)\psi$$

Aim: underline the similarities between these problems.

In the case $K(\rho) = 1/\rho$, and for irrotational flows, there is a formal equivalence between Euler-Korteweg and the nonlinear Schrödinger equation

$$i\partial_t\psi + \Delta\psi = g(|\psi|^2)\psi,$$

through the Madelung transform

$$(\rho,\ u = \nabla\phi) \to \psi := \sqrt{\rho}e^{i\phi/2}.$$

For technical reasons, this case simplifies a lot the analysis of the system, we focus here on the case of a general capillary coefficient.

Similarities appear both at the linear and nonlinear level.

Background for the Gross-Pitaevskii equation

- Conservation of energy $\frac{1}{2}\int_{\mathbb{R}^d} |\nabla\psi|^2 + ||\psi|^2 - 1|^2/2dx$, momentum $\mathrm{Im}\int \overline{\psi}\nabla\psi dx$.
- Global well-posedness in the energy space [Béthuel-Saut 99, Gérard 06, Killip et al 11].
- If $\psi_{in} = 1 + v_{in}$, v_{in} small enough the solution scatters in dimension ≥ 3 [Gustafson-Nakanishi-Tsai 08], existence of dispersive solutions in dimension 2
- Existence of travelling waves in any dimension [Béthuel et al 08, Maris 12] for speeds smaller than the “sound speed”. Some travelling waves are constructed as minimizers of the energy for a fixed momentum $P = \int \overline{\psi}\partial_1\psi$. Travelling waves that are not minimizers are known to exist too.
- Orbital stability of travelling waves in some cases [Chiron-Maris 17].

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Background for the Euler-Korteweg system

- Local well-posedness in $H^s \times (\rho_0 + H^{s+1})$, $s > d/2 + 1$, $\rho_0 \in \mathbb{R}^{+*}$, $\inf \rho_{in} > 0$. (+blow up criterion with a non vacuum condition on ρ) [BDD 06]
- In the case of quantum hydrodynamics, in dimension 2 and 3 existence of global weak solutions [Antonelli-Marcati 09].
- In the case of quantum hydrodynamics with $g(\rho) = \rho - \rho_0$ in dimension at least 3 and for small initial data, (EK) has global strong solutions [A-Haspot 14].
- Weak strong uniqueness [Giesselmann-Tzavaras]
- In dimension 1, existence of traveling waves, stability condition à la Grillakis-Shatah-Strauss. [Benzoni-Danchin-Descombes-Jamet 05]

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Two "hidden" structures

Nonlinear structure For u irrotational the system (EK) has an antisymetric structure

$$\partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} + \begin{pmatrix} 0 & \text{div} \\ \nabla & 0 \end{pmatrix} \begin{pmatrix} \delta H/\delta\rho \\ \delta H/\delta u \end{pmatrix} = 0. \tag{1}$$

with $H = \int_{\mathbb{R}^2} G(\rho) + \frac{1}{2}K(\rho)|\nabla\rho|^2 + \frac{1}{2}\rho|u|^2$, G primitive of g.
In particular, there is the energy conservation

$$\frac{d}{dt}H_{EK} = \frac{d}{dt}\int_{\mathbb{R}^d} G(\rho) + \frac{1}{2}K(\rho)|\nabla\rho|^2 + \frac{1}{2}\rho|u|^2 = 0.$$

Similarity with the Gross-Pitaevskii energy $\psi = \sqrt{\rho}e^{i\phi/2}$, $i\partial_t\psi + \Delta\psi = (|\psi|^2 - 1)\psi$:

$$\begin{aligned} H_{GP} &:= \frac{1}{2}\int_{\mathbb{R}^d} \frac{(\rho-1)^2}{2} + \frac{|\nabla\rho|^2}{4\rho} + \frac{\rho|u|^2}{4}dx \\ &= \frac{1}{2}\int_{\mathbb{R}^d} \frac{(|\psi|^2-1)^2}{2} + |\nabla\psi|^2 dx \end{aligned}$$

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Due to the invariance by translation of the equations, there is also conservation of the "momentum"

$$P(\rho) = \int (\rho - \rho_0) u \, dx,$$

where ρ_0 is a constant state such that $\lim_{\infty} \rho(x) = \rho_0$. The same is also true for Gross-Pitaevskii

$$P_{GP}(\rho, u) = \int_{\mathbb{R}^d} (\rho - 1) u \, dx = \int_{\mathbb{R}^d} \overline{\psi} \nabla \psi \, dx.$$

This analogy paves the way to the existence of traveling waves by constrained minimization.

Linear structure

For simplicity, consider $\rho = 1 + r$, $u = \nabla\phi$, the linearized system reads

$$\begin{cases} \partial_t r + \Delta\phi & = 0, \\ \partial_t \phi - K(1)\Delta r + g'(1) r & = 0. \end{cases}$$

Equivalently, setting $\widetilde{r} = \sqrt{K(1)}r$, $a = \sqrt{K(1)}$, $z = \widetilde{r} + i\phi$

$$\begin{cases} \partial_t \widetilde{r} + a\Delta\phi & = 0, \\ \partial_t \phi - a\Delta\widetilde{r} + g'(1)\sqrt{\frac{1}{K(1)}}\widetilde{r} & = 0, \end{cases}$$

$$\Leftrightarrow i\partial_t z + a\Delta z = g'(1)\sqrt{\frac{1}{K(1)}}\mathrm{Re}(z).\ (LEK)$$

If $g'(1) > 0$ this is (almost) the linearization of the Gross-Pitaevskii equation $i\partial_t\psi + \Delta\psi = (|\psi|^2 - 1)\psi$ near $\psi = 1$:

$$i\partial_t z + \Delta z = 2\mathrm{Re}(z).\ (LGP)$$

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Main results:

Theorem (A.-Haspot 16)

If $g'(\rho_0) > 0$, $d \geq 3$, for small, smooth, and irrotational initial data the local strong solution is global. Moreover, the solution scatters, i.e. converges to a solution of the linearized system as $t \to \infty$.

Theorem (A. 17)

With the same assumptions, in dimension 2, there exists travelling waves of arbitrarily small energy.

Remarks:

- Smallness is required for u_0, not its potential ϕ_0.
- The smallness and smoothness can be quantified.
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Sketch of proof of the second theorem

Based on variational methods, in the spirit of [Bethuel-Gravejat-Saut 08] for the Gross-Pitaevskii equation.

In the irrotational case $u = \nabla\phi$, a travelling wave is (up to symmetries) a solution of

$$\begin{cases} -c\partial_1\rho + \mathrm{div}(\rho\nabla\phi) = 0, \\ -c\partial_1\phi + |\nabla\phi|^2/2 + g(\rho) = K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2 \end{cases} \quad \text{(TW)}$$

which can be recast as

$$\begin{aligned} \delta H &= c\delta P, \\ \text{where} \quad H(\rho,\phi) &= \int_{\mathbb{R}^d} \frac{K(\rho)|\nabla\rho|^2 + \rho|\nabla\phi|^2}{2} + G(\rho)dx, \\ P(\rho,\phi) &= \int_{\mathbb{R}^d} (\rho-1)\partial_1\phi. \end{aligned}$$

I.e. a solitary wave can be seen as a critical point of $H - cP$ or a constrained minimizer of H with P constant.

We follow the constrained approach. Main issues:

- If the minimizer exists, its smoothness is not clear from the elliptic equation it satisfies,
- Even in the simple case $G(\rho) = (\rho - 1)^2/2$, the functional $H(\rho, \phi) = \int_{\mathbb{R}^d} \frac{K(\rho)|\nabla\rho|^2 + \rho|\nabla\phi|^2}{2} + G(\rho)dx$ is not coercive (and not even defined) on $H^1 \times \dot{H}^1$,
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Scheme of proof

- New coercive functional $\widetilde{H}$ with $\widetilde{H}(\rho,\phi) = H(\rho,\phi)$ if $\|\rho - 1\| << 1$.
- Constrained minimization of $\widetilde{H}$ on $\mathbb{R}^2/(2n\pi\mathbb{Z})^2$ instead of $\mathbb{R}^2$
- Smoothness of the minimizers (ρ_n, u_n), a priori estimates
- Concentration compactness argument on u_n

Minimization and smoothness for small data:

Existence of minimizers (on the torus) is easy. Smoothness can not be obtained directly:

$$u \in H^1, \Delta u = |\nabla u|^2 \Rightarrow \Delta u \in L^1 \Rightarrow u \in W^{2,1} \hookrightarrow H^1.$$

(and most likely not true: large traveling waves of GP have vortices)

For $\|r\|_{H^1} << 1$, model problem $\Delta r = |\nabla r|^2 + f$. Since $H^{1/2} \hookrightarrow L^4$ and interpolation

$$\begin{aligned} \|\Delta r\|_{L^2} &\leq \|\nabla r\|_{L^4}^2 + \|f\|_{L^2} \lesssim \|\nabla r\|_{L^2} \|\nabla^2 r\|_{L^2} + \|f\|_{L^2} \\ \Rightarrow \|\Delta r\|_{L^2} &\lesssim \|f\|_{L^2} + \|r\|_{H^1}. \end{aligned}$$

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A short glimpse of the concentration compactness argument:

- Set $H_{\min}(p) = \inf\{H(\rho, u) :\ P(\rho, u) = p\}$. Then $H_{\min}$ is strictly subadditive.
- Let (ρ_n, u_n) be a constrained minimizer on the torus $\mathbb{R}^2/(2\pi n\mathbb{Z})^2$, embed the torus in $\mathbb{R}^2$.
- Assume dichotomy occurs, i.e. there exists two profiles $(\rho^1, u^1),\ (\rho^2, u^2)$ such that

  $$H(\rho_n, u_n) \to H(\rho^1, u^1) + H(\rho^2, u^2),\ P(\rho_n, u_n) \to P(\rho^1, u^1) + P(\rho^2, u^2) := p_1 + p_2.$$

  One can prove

  $$\liminf H(\rho_n, u_n) \leq H_{\min}(p),$$

  thus

  $$H_{\min}(p_1) + H_{\min}(p_2) \leq H(\rho^1, u^1) + H(\rho^2, u^2) \leq H_{\min}(p)$$

  This contradicts the subadditivity.

Main issue is “spreading”, i.e.

$$u_n \to 0 \text{ unif. on } \Omega \text{ and } \|u_n\|_{H^1(\Omega)} \nrightarrow 0.$$

This is forbidden by the key estimate

$$\|\rho - 1\|_{L^\infty(K^c)} << 1 \Rightarrow \left| \int_{K^c} \widetilde{h}(\rho, \phi) - cp(\rho, \phi) dx \right| << \widetilde{h}(\rho, \phi).$$

to be understood as

“the energy spreads similarly to the momentum”

Some (very open) perspectives:

- solitary waves in higher dimensions, stability
- more precisions on their minimal energy,
- Global existence in dimension one,
- finite time blow up.

Thank you for your attention