A Well Posed Coupling Procedure of the Compressible and Incompressible Navier-Stokes Equations

Fatemeh Ghasemi & Jan Nordström

Division of Computational Mathematics, Department of Mathematics,
Linköping University, SE-581 83 Linköping, Sweden

COUPLED PROBLEMS 2017

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- Interface conditions
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Introduction and motivation

If a well posed problem does not exist:

- An accurate numerical approximation can be made.
- A stable numerical approximation can be made.
- An accurate *and* stable approximation can *not* be made.
- Well-posedness is the *most important* point in coupling procedures.
- Once well-posedness is established, stability follows almost automatically by using the SBP-SAT technique.
- In this talk we focus on well-posedness, and its link to stability.

The compressible Navier-Stokes equations

The linearized and symmetrized compressible Navier-Stokes equations are:

$$U_t + A_1 U_x + A_2 U_y = \epsilon\big((A_{11} U_x + A_{12} U_y)_x + (A_{21} U_x + A_{22} U_y)_y\big) \tag{1}$$

where

$$U = \left[\frac{\bar{c}\rho}{\sqrt{\gamma}\bar{\rho}}, u, v, \frac{T}{\bar{c}\sqrt{\gamma(\gamma-1)}}\right]^T.$$

The coefficient matrices are:

$$A_1 = \begin{bmatrix} \bar{u} & \bar{c}/\sqrt{\gamma} & 0 & 0 \\ \bar{c}/\sqrt{\gamma} & \bar{u} & 0 & \bar{c}\sqrt{\frac{\gamma-1}{\gamma}} \\ 0 & 0 & \bar{u} & 0 \\ 0 & \bar{c}\sqrt{\frac{\gamma-1}{\gamma}} & 0 & \bar{u} \end{bmatrix}, A_2 = \begin{bmatrix} \bar{v} & 0 & \bar{c}/\sqrt{\gamma} & 0 \\ 0 & \bar{v} & 0 & 0 \\ \bar{c}/\sqrt{\gamma} & 0 & \bar{v} & \bar{c}\sqrt{\frac{\gamma-1}{\gamma}} \\ 0 & 0 & \bar{c}\sqrt{\frac{\gamma-1}{\gamma}} & \bar{v} \end{bmatrix}.$$

The compressible Navier-Stokes equations

$$A_{11} = \frac{1}{\bar{\rho}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda + 2\mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \theta \end{bmatrix}, \quad A_{22} = \frac{1}{\bar{\rho}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \lambda + 2\mu & 0 \\ 0 & 0 & 0 & \theta \end{bmatrix},$$

$$A_{12} = A_{21}^T = \frac{1}{\bar{\rho}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The incompressible Navier-Stokes equations

The linearized incompressible Navier-Stokes equations are:

$$\begin{aligned}
\tilde{u}_t + \hat{u}\tilde{u}_x + \hat{v}\tilde{u}_y =& \frac{1}{\hat{\rho}}(-\tilde{p}_x + \epsilon\tilde{\mu}(\tilde{u}_{xx} + \tilde{u}_{yy})), \\
\tilde{v}_t + \hat{u}\tilde{v}_x + \hat{v}\tilde{v}_y =& \frac{1}{\hat{\rho}}(-\tilde{p}_y + \epsilon\tilde{\mu}(\tilde{v}_{xx} + \tilde{v}_{yy})), \\
\tilde{u}_x + \tilde{v}_y =& 0.
\end{aligned}$$

They can be rewritten, using $(\tilde{u}_x + \tilde{v}_y)_x = (\tilde{u}_x + \tilde{v}_y)_y = 0$, as

$$\tilde{I}V_t + B_1 V_x + B_2 V_y = \epsilon((B_{11}V_x + B_{12}V_y)_x + (B_{21}V_x + B_{22}V_y)_y) \quad (2)$$

where

$$V = \left[\, \tilde{u}, \tilde{v}, \tilde{p} \,\right]^T.$$

Note that (1) and (2) have the same form.

The incompressible Navier-Stokes equations

The coefficient matrices are:

$$\tilde{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} \hat{u} & 0 & 1/\hat{\rho} \\ 0 & \hat{u} & 0 \\ 1/\hat{\rho} & 0 & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} \hat{v} & 0 & 0 \\ 0 & \hat{v} & 1/\hat{\rho} \\ 0 & 1/\hat{\rho} & 0 \end{bmatrix}, \quad B_{11} = \frac{1}{\hat{\rho}} \begin{bmatrix} 2\tilde{\mu} & 0 & 0 \\ 0 & \tilde{\mu} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B_{22} = \frac{1}{\hat{\rho}} \begin{bmatrix} \tilde{\mu} & 0 & 0 \\ 0 & 2\tilde{\mu} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_{12} = B_{21}^T = \frac{1}{\hat{\rho}} \begin{bmatrix} 0 & 0 & 0 \\ \tilde{\mu} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The energy method (multiplying the equations by U^T and V^T respectively, and integrating over the spatial domains) leads to

$$\frac{d}{dt}(\|U\|_2^2 + \alpha_c \|V\|_{\tilde{I}}^2) + 2\epsilon D_1 + 2\epsilon D_2 = -\int_0^1 W^T E W dx \qquad (3)$$

where $\alpha_c > 0$ is a free parameter,

$$E = \begin{bmatrix} -A_2 & \epsilon \tilde{I}_4 & 0 & 0 \\ \epsilon \tilde{I}_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon \tilde{I} \\ 0 & 0 & -\epsilon \tilde{I} & B_2 \end{bmatrix}, \quad \tilde{I}_4 = diag\,(0,1,1,1)\,,$$

and

$$W = [U, A_{21}U_x + A_{22}U_y, B_{21}V_x + B_{22}V_y, V]^T, \quad D_1 \geq 0, \quad D_2 \geq 0.$$

We ignore the boundary conditions and focus only on the interface.

The matrix E has

- 5 five positive eigenvalues
- 4 zero eigenvalues
- five negative eigenvalues

$$\Downarrow$$

5 interface conditions are needed.

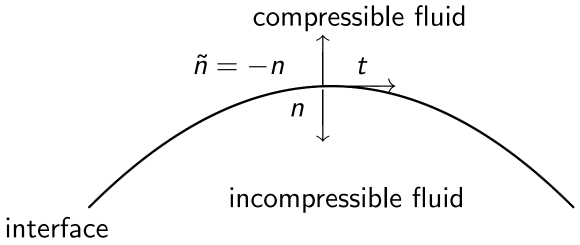

Figure: A sketch of a fluid-fluid interface.

Interface conditions

The form of interface conditions

- Continuity of velocity of the fluids at the interface require

$$u = \tilde{u}, \quad v = \tilde{v}.$$

- The stresses exerted on the interface by the compressible and the incompressible fluid are equal.

$$\sigma n = -\tilde{\sigma}\tilde{n},$$

where

$$\sigma = pI_2 - \epsilon\tau, \quad \tilde{\sigma} = \tilde{p}I_2 - \epsilon\tilde{\tau}.$$

  - Tangential interface condition:

$$t^T \tau n = t^T \tilde{\tau} n.$$

  - Normal interface condition:

$$p - \epsilon n^T \tau n = \tilde{p} - \epsilon n^T \tilde{\tau} n.$$

By inserting the coupling conditions into (3) we find

$$\frac{d}{dt}(\|U\|_2^2+\|V\|_{\tilde{I}}^2) + 2\epsilon D_1 + 2\epsilon D_2 = (\frac{1}{\bar{\rho}} - \frac{\alpha_c}{\hat{\rho}}) \int_0^1 ITdx$$

$$+\frac{2\epsilon\kappa}{\bar{\rho}\bar{c}^2(\gamma-1)c_p} TT_y.$$

Choosing $\alpha_c = \hat{\rho}/\bar{\rho}$ leads to

$$\frac{d}{dt}(\|U\|_2^2 + \|V\|_{\tilde{I}}^2) + 2\epsilon D_1 + 2\epsilon D_2 = \frac{2\epsilon\kappa}{\bar{\rho}\bar{c}^2(\gamma-1)c_p} TT_y.$$

$\Longrightarrow$ one more condition is needed, as previously indicated.

We add on the decoupled heat equation for the incompressible fluid

$$\tilde{T}_t + \hat{u}\tilde{T}_x + \hat{v}\tilde{T}_y = \frac{\epsilon\tilde{\theta}}{\hat{\rho}}\Delta\tilde{T}, \quad \tilde{\theta} = \frac{\tilde{\kappa}}{\tilde{c_p}Pe}.$$

By adding the heat equation, six interface conditions are needed.
We use the continuity of temperature and fluxes across the interface

$$T = \tilde{T}, \quad \kappa T_y = \tilde{\kappa}\tilde{T}_y.$$

The energy method leads to

$$\frac{d}{dt}(\|U\|_{H_1}^2 + \alpha_c\|V\|_{H_2}^2) + 2\epsilon D_1 + 2\epsilon D_2 = TT_y\frac{2\kappa\epsilon}{\bar{\rho}}(\frac{\delta_1}{\bar{c}^2(\gamma-1)c_p} - \frac{\delta_2}{\tilde{c_p}Pe}),$$

where

$$\|U\|_{H_1}^2 = \int_{\Omega_1} U^T H_1 U d\Omega, \ \|V\|_{H_2}^2 = \int_{\Omega 2} V^T H_2 V d\Omega,$$

and

$$H_1 = diag(1,1,1,\delta_1), \quad H_2 = diag(1,1,0,\delta_2), \quad \delta_{1,2} > 0.$$

Well-posedness

The energy method

Choosing

$$\alpha_c = \hat{\rho}/\bar{\rho}, \quad \delta_2 = \frac{\tilde{c_p} Pe \delta_1}{\bar{c}^2(\gamma - 1)c_p},$$

leads to

$$\frac{d}{dt}(\|U\|^2_{H_1} + \alpha_c \|V\|^2_{H_2}) + 2\epsilon D_1 + 2\epsilon D_2 = 0,$$

which means that energy is bounded. Note that δ_1 is arbitrary.

The energy bound leads directly to uniqueness and since we use a minimal number of interface conditions, we have existence and consequently well-posdeness.

The semi-discrete problem

SBP-SAT technique

SBP operators

$$
\begin{aligned}
U_x &\approx (D_x \otimes I_4)\mathbf{U}, & V_x &\approx (\tilde{D}_x \otimes I_4)\mathbf{V}, & & \\
U_y &\approx (D_y \otimes I_4)\mathbf{U}, & V_y &\approx (\tilde{D}_y \otimes I_4)\mathbf{V}, & & \\
D_x &= P_x^{-1} Q_x \otimes I_y, & \tilde{D}_x &= \tilde{P}_x^{-1} \tilde{Q}_x \otimes \tilde{I}_y, & P_{x,y} &= P_{x,y}^T > 0, \\
D_y &= P_y^{-1} Q_y \otimes I_x, & \tilde{D}_y &= \tilde{P}_y^{-1} \tilde{Q}_y \otimes \tilde{I}_x, & \tilde{P}_{x,y} &= \tilde{P}_{x,y}^T > 0,
\end{aligned}
$$

$$
\begin{aligned}
Q_{x,y} + Q_{x,y}^T &= diag(-1, 0, ..., 0, 1), \\
\tilde{Q}_{x,y} + \tilde{Q}_{x,y}^T &= diag(-1, 0, ..., 0, 1).
\end{aligned}
$$

The semi-discrete SBP-SAT approximation is

$$\mathbf{U}_t + [D_x \otimes A_1 + D_y \otimes A_2]\mathbf{U} = \epsilon(D_x \otimes I_4)[D_x \otimes A_{11} + D_y \otimes A_{12}]\mathbf{U}$$
$$+\epsilon(D_y \otimes I_4)[D_x \otimes A_{21} + D_y \otimes A_{22})]\mathbf{U} + \mathbb{S},$$

$$\mathbf{I}\mathbf{V}_t + [\tilde{D}_x \otimes B_1 + \tilde{D}_y \otimes B_2]\mathbf{V} = \epsilon(\tilde{D}_x \otimes I_4)[\tilde{D}_x \otimes B_{11} + \tilde{D}_y \otimes B_{12}]\mathbf{V}$$
$$+\epsilon(\tilde{D}_y \otimes I_4)[\tilde{D}_x \otimes B_{21} + \tilde{D}_y \otimes B_{22}]\mathbf{V} + \tilde{\mathbb{S}},$$

where

$$\mathbf{I} = \tilde{I}_x \otimes \tilde{I}_y \otimes \tilde{I}.$$

The penalty terms are

$$\mathbb{S} = (I_x \otimes P_y^{-1} E \otimes \Sigma)\big(C\mathbf{U} - \tilde{C}\mathbf{V}\big), \quad \tilde{\mathbb{S}} = (\tilde{I}_x \otimes \tilde{P}_y^{-1} \tilde{E} \otimes \tilde{\Sigma})\big(\tilde{C}\mathbf{V} - C\mathbf{U}\big),$$

where

$$\begin{aligned} C &= I_x \otimes I_y \otimes C_1 - D_x \otimes C_2 - D_y \otimes C_3, \\ \tilde{C} &= \tilde{I}_x \otimes \tilde{I}_y \otimes \tilde{C}_1 - \tilde{D}_x \otimes \tilde{C}_2 - \tilde{D}_y \otimes \tilde{C}_3, \end{aligned}$$

and $C_1 U - C_2 U_x - C_3 U_y = \tilde{C}_1 V - \tilde{C}_2 V_x - \tilde{C}_3 V_y$ is the matrix form of interface conditions.

Stability

Applying the discrete energy method and choosing $\alpha_d = \alpha_c$ and the penalty matrices

$$\Sigma = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\bar{\rho} & 0 & 0 \\ 0 & 0 & \epsilon/\bar{\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \tilde{\Sigma} = \begin{bmatrix} 0 & 0 & 0 & -1/\hat{\rho} & 0 & 0 \\ 0 & 0 & -\epsilon/\hat{\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

and δ_2 as before, leads to

$$\frac{d}{dt}(\|\mathbf{U}\|^2_{J_1} + \alpha_d \|\mathbf{V}\|^2_{J_2}) + 2\epsilon \mathbf{D}_1 + 2\epsilon \mathbf{D}_2 = 0,$$

where

$$J_1 = I_x \otimes I_y \otimes H_1 \quad J_2 = \tilde{I}_x \otimes \tilde{I}_y \otimes H_2.$$

This means that the semi-discrete problem is stable.

Thank you for listening!