Discrete transparent boundary conditions for the mixed KdV-BBM equation

David Sanchez

Joint work with Pascal Noble and Christophe Besse

Institut de Mathématiques de Toulouse

NABUCO

Talk's content

Context

Water waves models

- Dispersive regularization of hyperbolic conservation laws
- Dispersive shock waves:
  - oscillatory structure,
  - the width of the socillatory region grows with time.

Numerical simulations are difficult

- Rankine-Hugoniot jump conditions not satisfied
- Spectral techniques:
  + suitable to describe oscillatory phenomena,
  - periodic boundary conditions,
  - very large domains for long time simulations,
  - dynamic of dispersive equations very different in periodic domain and in the whole space.

Context

Water waves models

- Dispersive regularization of hyperbolic conservation laws
- Dispersive shock waves:
  - oscillatory structure,
  - the width of the socillatory region grows with time.

Transparent boundary conditions

- adapted to simulations for the whole space domain,
- The solution calculated in the computational domain is an approximation of the exact solution restricted to the computational domain.

Equations

Korteweg de Vries equation

$$\partial_t u + \partial_x u + \frac{3\eta}{2} u\partial_x u + \frac{\mu}{6}\partial_{xxx} u = 0, \quad \forall t > 0, \quad \forall x \in \mathbb{R}.$$

As η and $\mu \to 0$ we have

$$\partial_x u = -\partial_t u + O(\eta + \mu)$$

One can trade a spatial derivative for a time derivative

KdV-BBM equation

$$\partial_t \left(u - \alpha\partial_{xx} u\right) + \partial_x u + \frac{3\eta}{2} u\partial_x u + (\frac{\mu}{6} - \alpha)\partial_{xxx} u = 0, \quad \forall\, 0 < \alpha \leq \frac{\mu}{6}.$$

Equations

KdV-BBM equation

$$\partial_t \left(u - \alpha \partial_{xx} u\right) + \partial_x u + \frac{3\eta}{2} u \partial_x u + \left(\frac{\mu}{6} - \alpha\right)\partial_{xxx} u = 0, \quad \forall\, 0 < \alpha \leq \frac{\mu}{6}.$$

When $\alpha = \mu/6$, we have the

Benjamin-Bona-Mahoney equation

$$\partial_t \left(u - \alpha \partial_{xx} u\right) + \partial_x u + \frac{3\eta}{2} u \partial_x u = 0, \quad \forall t > 0, \quad \forall x \in \mathbb{R}.$$

Equations

Korteweg de Vries equation

$$\partial_t u + \partial_x u + \frac{3\eta}{2} u \partial_x u + \frac{\mu}{6} \partial_{xxx} u = 0, \quad \forall t > 0, \quad \forall x \in \mathbb{R}.$$

KdV-BBM equation

$$\partial_t (u - \alpha \partial_{xx} u) + \partial_x u + \frac{3\eta}{2} u \partial_x u + (\frac{\mu}{6} - \alpha) \partial_{xxx} u = 0, \quad \forall\, 0 < \alpha \leq \frac{\mu}{6}.$$

- solitary waves and cnoidal (periodic) waves solutions for these equations
- interaction between these waves
- role in the description of the solutions for asymptotically large time.

Equations

We focus on KdV-BBM linearized about a constant state $u = U$. This yields

linearized KdV-BBM equation

$$\partial_t(u - \alpha\partial_{xx}u) + c\partial_x u + \varepsilon\partial_{xxx}u = 0, \quad \forall t > 0, \quad \forall x \in \mathbb{R},$$

- dispersion parameters: α, ε
- velocity: $c = (1 + 3\eta\frac{U}{2})$

State of the art

- Schrödinger equation: discrete artifical boundary conditions:
  - Arnorld, Ehrhardt, Sofronov (2003),
  - Arnorld, Ehrhardt, Schulte, Sofronov (2012),
  - Ehrhardt (2001,2008),
  - Ehrhardt, Arnold (2001)
- Pure BBM case, $\varepsilon = 0$: continuous and discrete transparent boundary conditions, Besse, Mésognon-Giraud, Noble (2016)
- Pure KdV case, $\alpha = 0$:
  - continuous TBC, Zheng (2006), Zheng, Wen, Han (2008)
  - exact transparent and discrete boundary conditions, Besse, Ehrhardt, Lacroix-Violet (2016)

State of the art

In Besse, Ehrhardt, Lacroix-Violet (2016)

- DTBC derived for an upwind (first order) and a centered (second order) spatial discretization, time discretization based on the Crank-Nicolson scheme.
- DTBC perfectly adapated to the scheme, retain the stability property of the discretization method
- no reflexion when compared to the discrete whole space solution.
- In the case of the linearized KdV equation
  - Not explicit
  - Requires the numerical inversion of the $\mathcal{Z}$-transformation (discrete analogue of the inverse Laplace transform)
  - Numerical error and instabilities for large time simulations, Arnorld, Ehrhardt, Sofronov (2003), Zisowsky (2003)

Initial boundary value problem

$$
\begin{aligned}
\partial_t(u - \alpha\partial_{xx}u) + c\partial_x u + \varepsilon\,\partial_{xxx}u = 0, \quad \forall t > 0, \quad \forall x \in \mathbb{R}, \\
u(0,x) = u_0(x), \quad \forall x \in \mathbb{R}, \\
\lim_{x\to\infty} u(t,x) = \lim_{x\to-\infty} u(t,x) = 0,
\end{aligned}
$$

where

- u_0 is compactly supported in a finite computational interval $[x_\ell,\, x_r]$ with $x_\ell < x_r$,
- $c \in \mathbb{R}$ and $\alpha, \varepsilon > 0$ are respectively a velocity and two dispersion parameters.

Continuous artificial boundary condition problem

Outside $[x_\ell, x_r]$, we rewrite the equation as

$$\partial_x \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\varepsilon^{-1}\partial_t & -\varepsilon^{-1}\, c & \alpha\varepsilon^{-1}\partial_t \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

We use the Laplace transform with respect to time. Then for all $s \in \mathbb{C}$ with $\Re(s) > 0$:

$$\partial_x \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\varepsilon^{-1}\, s & -\varepsilon^{-1}\, c & \alpha\varepsilon^{-1} s \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{pmatrix} := \mathcal{A}_{\alpha,\varepsilon}(s,c) \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{pmatrix}.$$

Continuous artificial boundary condition problem

The general solutions of this system of ODE are given explicitly by

$$\begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{pmatrix} = e^{\lambda_1(s)\,x}\,\mathcal{V}_1(s) + e^{\lambda_2(s)\,x}\,\mathcal{V}_2(s) + e^{\lambda_3(s)\,x}\,\mathcal{V}_3(s), \quad x < x_\ell, \quad x > x_r,$$

where $\lambda_k(s), k = 1, 2, 3$ are the roots of

$$P(s, c, \alpha, \varepsilon, \lambda) = s + c\lambda - \alpha s \lambda^2 + \varepsilon \lambda^3 = 0$$

and $\mathcal{V}_k = \left(1, \lambda_k, \lambda_k^2\right)^T$ are the right eigenvectors of the matrix $\mathcal{A}_{\alpha,\varepsilon}(s, c)$ associated to the eigenvalue λ_k.

Continuous artificial boundary condition problem

Proposition

For all $\varepsilon > 0$ and for all $\alpha \geq 0$, the roots $\lambda_k(s), k = 1, 2, 3$ possess the following separation property:

$$\Re(\lambda_1(s)) < 0, \quad \Re(\lambda_2(s)) > 0, \quad \Re(\lambda_3(s)) > 0, \quad \forall \Re(s) > 0.$$

Sketch of the proof:

- The property has been proved if $\alpha = 0$ in Besse, Ehrhardt, Lacroix-Violet (2016).
- Continuity argument for the number of roots with a positive real part.

Continuous artificial boundary condition problem

Now, we search for solutions $(\hat{u}, \hat{v}, \hat{w})^T$ such that $\lim_{x \to \infty} \hat{u}(s, x) = 0$. It is satisfied provided that we impose the condition

$$\mathcal{V}_1(s) \wedge \begin{pmatrix} \hat{u}(s, x_r) \\ \hat{v}(s, x_r) \\ \hat{w}(s, x_r) \end{pmatrix} = 0,$$

which in turn provides the following two boundary conditions

$$\partial_x \hat{u}(s, x_r) = \lambda_1(s) \hat{u}(s, x_r), \qquad \partial_{xx} \hat{u}(s, x_r) = \lambda_1^2(s) \hat{u}(s, x_r).$$

Continuous artificial boundary condition problem

A similar argument to obtain solutions $(\hat{u}, \hat{v}, \hat{w})^T$ such that $\lim_{x\to-\infty} \hat{u}(s,x) = 0$. We therefore have to impose the condition

$$\mathcal{V}_2(s) \wedge \mathcal{V}_3(s) \cdot \begin{pmatrix} \hat{u}(s, x_\ell) \\ \hat{v}(s, x_\ell) \\ \hat{w}(x, x_\ell) \end{pmatrix} = 0,$$

which gives the following boundary condition.

$$\partial_{xx}\hat{u}(s, x_\ell) - (\lambda_2(s) + \lambda_3(s))\partial_x \hat{u}(s, x_\ell) + \lambda_2\lambda_3\hat{u}(s, x_\ell) = 0.$$

Continuous artificial boundary condition problem

A similar argument to obtain solutions $(\hat{u}, \hat{v}, \hat{w})^T$ such that $\lim_{x\to-\infty} \hat{u}(s,x) = 0$. We therefore have to impose the condition

$$\mathcal{V}_2(s) \wedge \mathcal{V}_3(s) \cdot \begin{pmatrix} \hat{u}(s,x_\ell) \\ \hat{v}(s,x_\ell) \\ \hat{w}(x,x_\ell) \end{pmatrix} = 0,$$

which gives the following boundary condition.

$$\begin{aligned} \partial_{xx}\hat{u}(s,x_\ell) + \left(\lambda_1(s) - \frac{\alpha\, s}{\varepsilon}\right)\partial_x \hat{u}(s,x_\ell) \\ + \left(\lambda_1(s)^2 - \frac{\alpha\, s}{\varepsilon}\lambda_1(s) + \frac{c}{\varepsilon}\right)\hat{u}(s,x_\ell) &= 0. \end{aligned}$$

Continuous artificial boundary condition problem

Written in time variables, the boundary conditions read

$$
\begin{aligned}
&\partial_x u(t,x_r) = \mathcal{L}^{-1}(\lambda_1(s)) * u(t,x_r),\\
&\partial_{xx} u(t,x_r) = \mathcal{L}^{-1}(\lambda_1^2(s)) * u(t,x_r),\\
&\partial_{xx} u(t,x_\ell) + \mathcal{L}^{-1}(\lambda_1(s) - \frac{\alpha\, s}{\varepsilon}) * \partial_x u(t,x_\ell)\\
&+\mathcal{L}^{-1}(\lambda_1(s)^2 - \frac{\alpha\, s}{\varepsilon}\lambda_1(s)) * u(t,x_\ell) + \frac{c}{\varepsilon} u(t,x_\ell) = 0.
\end{aligned}
$$

Continuous artificial boundary condition problem

Proposition

Assume that

$$\frac{c}{2} + \varepsilon \left(\Re(\lambda_1^2(i\xi)) - \frac{|\lambda_1(i\xi)|^2}{2} \right) - \alpha \Re(i\xi\lambda_1(i\xi)) \geq 0, \quad \forall \xi \in \mathbb{R}.$$

Then the problem

$$\begin{cases} \partial_t(u - \alpha\partial_{xx}u) + c\partial_x u + \varepsilon\,\partial_{xxx}u = 0, & (t,x) \in \mathbb{R}_*^+ \times (x_\ell, x_r), \\ u(0,x) = u_0(x), & x \in (x_\ell, x_r), \end{cases}$$

with the previous boundary conditions is H^1-stable. For any $t > 0$,

$$\int_{x_\ell}^{x_r} u^2(t,x) + \alpha(\partial_x u)^2(t,x)\,dx \leq \int_{x_\ell}^{x_r} u_0^2(x) + \alpha(\partial_x u_0)^2\,dx.$$

Continuous artificial boundary condition problem

Sketch of the proof:

- The root $\lambda_1(s)$ is defined for all $s \in \mathbb{C}$ such that $\Re(s) > 0$. We define $\lambda_1(i\xi)$ with $\xi \in \mathbb{R}$ as

$$\lambda_1(i\xi) = \lim_{\eta \to 0^+} \lambda_1(\eta + i\xi).$$

- $\mathcal{E}(t) = \mathcal{E}(0) + J(x_\ell) - J(x_r)$ with

$$J(x) = \int_0^t (cu^2 + 2\varepsilon u \partial_{xx}^2 u - \varepsilon(\partial_x u)^2 - 2\alpha u \partial_{xt}^2 u)(t,x)\, dt$$

- We let $U = u(t, x_\ell)\mathbf{1}_{[0,T]}$ and $V = \partial_x u(t, x_\ell)\mathbf{1}_{[0,T]}$ on the left hand side, $U = u(t, x_r)\mathbf{1}_{[0,T]}$ on the right hand side to rewrite the integrals in the form $\displaystyle\int_0^{+\infty} \dots dt$ and use the boundary conditions to get the result.

Continuous artificial boundary condition problem

Proposition

The stability condition given in Prop. 2 is always fulfilled:

$$\forall \xi \in \mathbb{R}, \quad \frac{c}{2} + \varepsilon \left(\Re(\lambda_1^2(i\xi)) - \frac{|\lambda_1(i\xi)|^2}{2} \right) - \alpha \Re(i\xi\lambda_1(i\xi)) \geq 0.$$

Sketch of the proof: study of all the cases obtained by writing $\lambda_1(i\xi) = a + ib$.

Ideas

- Not possible to compute explicitly the inverse Laplace transform of λ_k, $k = 1, 2, 3$,
  - $\Rightarrow$ no closed form of the boundary conditions.
  - $\Rightarrow$ difficult to discretize the transparent boundary conditions
- Construction of the transparent boundary conditions on the fully discrete scheme.
- $\mathcal{Z}$ transform instead of the Laplace transform
- As in continuous case, explicit inverse $\mathcal{Z}$ transform is not available. $\Rightarrow$ heavy numerical cost (see Besse, Ehrhardt, Lacroix-Violet (2016))
- Alternative approach to construct "explicit" coefficients of discrete kernels.

Numerical scheme for linear KdV-BBM equation

$$
\begin{aligned}
u_j^{n+1} - u_j^n \quad & - \quad \lambda_B \left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1} - u_{j+1}^n + 2u_j^n - u_{j-1}^n \right) \\
& + \quad \frac{\lambda_H}{4} \left(u_{j+1}^{n+1} - u_{j-1}^{n+1} \right) + \frac{\lambda_H}{4} \left(u_{j+1}^n - u_{j-1}^n \right) \\
& + \quad \frac{\lambda_D}{4} \left(u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2\, u_{j-1}^{n+1} - u_{j-2}^{n+1} \right) \\
& + \quad \frac{\lambda_D}{4} \left(u_{j+2}^n - 2u_{j+1}^n + 2\, u_{j-1}^n - u_{j-2}^n \right) = 0, \ \ \forall j = 0, \dots, J,
\end{aligned}
$$

with

$$
\lambda_H = \frac{c\delta t}{\delta x}, \quad \lambda_D = \frac{\varepsilon \delta t}{\delta x^3}, \quad \lambda_B = \frac{\alpha}{\delta x^2},
$$

δt the time step, δx the space step, $J = (x_r - x_\ell)/\delta x$, u_j^n the approximation of the exact solution $u(t, x)$ at points $j\delta x$ and instants $n\delta t$

$\mathcal{Z}-$transform of the equation

$$\hat{u}(z) = \mathcal{Z}\{(u^n)_n\}(z) = \sum_{k=0}^{\infty} u^k \, z^{-k}, \quad |z| > R > 0,$$

where R is the convergence radius of the Laurent series and $z \in \mathbb{C}$. Denoting $\hat{u}_j = \hat{u}_j(z)$ the $\mathcal{Z}-$transform of the sequence $(u_j^{(n)})_{n\in\mathbb{N}}$, we obtain the homogeneous *fourth order difference equation*

$$\begin{aligned}
\hat{u}_{j+2} \quad & - \quad \left(2 - \frac{\lambda_H}{\lambda_D} + \frac{4\lambda_B}{\lambda_D}\frac{z-1}{z+1}\right)\hat{u}_{j+1} \\
& + \quad \left(\frac{4}{\lambda_D} + \frac{8\lambda_B}{\lambda_D}\right)\frac{z-1}{z+1}\hat{u}_j \\
& + \quad \left(2 - \frac{\lambda_H}{\lambda_D} - \frac{4\lambda_B}{\lambda_D}\frac{z-1}{z+1}\right)\hat{u}_{j-1} - \hat{u}_{j-2} = 0.
\end{aligned}$$

$\mathcal{Z}-$transform of the equation

$$\hat{u}(z) = \mathcal{Z}\{(u^n)_n\}(z) = \sum_{k=0}^{\infty} u^k\, z^{-k}, \quad |z| > R > 0,$$

where R is the convergence radius of the Laurent series and $z \in \mathbb{C}$.
Associated characteristic polynomial:

$$P(r) = r^4 - (2-a+\mu p(z))r^3 + \left(\frac{4a}{\lambda_H} + 2\mu\right) p(z) r^2 + (2-a-\mu p(z))r - 1 = 0.$$

with

$$a = \frac{\lambda_H}{\lambda_D}, \quad \mu = \frac{4\lambda_B}{\lambda_D}, \quad p(z) = \frac{z-1}{z+1} = \frac{1-z^{-1}}{1+z^{-1}}.$$

$\mathcal{Z}-$transform of the equation

Proposition

Assume $\varepsilon > 0, \alpha \geq 0$, $\delta x, \delta t > 0$ and $c \in \mathbb{R}$. Then, the roots of P are well separated according to

$$|r_1(z)| < 1, \quad |r_2(z)| < 1, \qquad |r_3(z)| > 1, \quad |r_4(z)| > 1$$

which defines the discrete separation properties. As a consequence, there is a smooth parameterization of the "stable" (respectively "unstable") subspace $\mathbb{E}^s(z)$ (resp $\mathbb{E}^u(z)$) of solutions to (8) which decrease to 0 as $j \to +\infty$ (respectively $j \to -\infty$) for $|z| > R$ with R large enough.

Discrete transparent boundary conditions

According to this proposition, we set

$$S^s(z) = r_1(z) + r_2(z),\ P^s(z) = r_1(z)r_2(z),$$

$$S^u(z) = r_3(z) + r_4(z),\ P^u(z) = r_3(z)r_4(z)$$

and the characteristic polynomial P admits the factorization

$$P(r) = \left(r^2 - S^u(z)r + P^u(z)\right)\left(r^2 - S^s(z)r + P^s(z)\right)$$

Discrete transparent boundary conditions

The discrete transparent boundary conditions are written as follows. On the left boundary, one must have

$$(\hat{u}_{-2}, \hat{u}_{-1}, \hat{u}_0, \hat{u}_1) \in \mathbb{E}^u(z)$$

which is also equivalent to the following boundary conditions

$$\hat{u}_1 - S^u(z)\,\hat{u}_0 \,+ P^u(z)\,u_{-1} = 0,$$

$$\hat{u}_0 - S^u(z)\,\hat{u}_{-1} \,+ P^u(z)\,u_{-2} = 0.$$

Discrete transparent boundary conditions

The discrete transparent boundary conditions are written as follows. On the other hand, one must have on the right boundary

$$(\hat{u}_{J-1}, \hat{u}_J, \hat{u}_{J+1}, \hat{u}_{J+2}) \in \mathbb{E}^s(z)$$

which is also written as

$$\hat{u}_{J+2} - S^s(z)\,\hat{u}_{J+1} + P^s(z)\,\hat{u}_J = 0,$$

$$\hat{u}_{J+1} - S^s(z)\,\hat{u}_J + P^s(z)\,\hat{u}_{J-1} = 0.$$

Discrete transparent boundary conditions

The coefficients of P admits a singularity at $z=-1$

- $\Rightarrow$ bad behavior of the coefficients in the expansion of S^u, P^u, S^s, P^s.
- $\Rightarrow$ Alternative boundary conditions by multiplying $1+z^{-1}$.

Inverting the $\mathcal{Z}-$transform, one finds that the left and right boundary conditions are written as:

$$u_1^{n+1} + u_1^n + \tilde{s}^u *_d \; u_0^{n+1} + \tilde{p}^u *_d \; u_{-1}^{n+1} = 0,$$

$$u_0^{n+1} + u_0^n + \tilde{s}^u *_d \; u_{-1}^{n+1} + \tilde{p}^u *_d \; u_{-2}^{n+1} = 0,$$

$$u_{J+2}^{n+1} + u_{J+2}^n + \tilde{s}^s *_d \; u_{J+1}^{n+1} + \tilde{p}^s *_d \; u_J^{n+1} = 0,$$

$$u_{J+1}^{n+1} + u_{J+1}^n + \tilde{s}^u *_d \; u_J^{n+1} + \tilde{p}^u *_d \; u_{J-1}^{n+1} = 0,$$

Discrete transparent boundary conditions

where the sequences $\tilde{S}^u, \tilde{P}^u$ and $\tilde{S}^s, \tilde{P}^s$ are defined as

$$\tilde{S}^s(z) = (1 + z^{-1}) S^s(z) = \sum_{n=0}^{\infty} \frac{\tilde{s}_n^s}{z^n},$$

$$\tilde{P}^s(z) = (1 + z^{-1}) P^s(z) = \sum_{n=0}^{\infty} \frac{\tilde{p}_n^s}{z^n},$$

$$\tilde{S}^u(z) = (1 + z^{-1}) S^u(z) = \sum_{n=0}^{\infty} \frac{\tilde{s}_n^u}{z^n},$$

$$\tilde{P}^u(z) = (1 + z^{-1}) P^u(z) = \sum_{n=0}^{\infty} \frac{\tilde{p}_n^u}{z^n}.$$

Discrete transparent boundary conditions

Computation of the coefficients

if one set $V(z) = \sum_{k=0}^{\infty} v_k\, z^{-k}$ for all $|z| > R$, the coefficients v_k are recovered by the formula

$$v_n = \frac{r^n}{2\pi} \int_0^{2\pi} V(r\, e^{i\phi}) e^{in\phi} d\phi, \quad \forall n \in \mathbb{N},$$

for some $r > R$ and the approximation of these integrals are done by using the Fast Fourier Transform.

Problem

For Schrödinger and lKdV equation, $R = 1$. Numerical procedure is instable as $n \to +\infty$

Discrete transparent boundary conditions

Let $x = 1/z$

Relation between coefficients and roots

$$
\begin{aligned}
&S^s(x) + S^u(x) = 2 - a + \mu\frac{1-x}{1+x},\\
&P^u(x) + P^s(x) + S^u(x)S^s(x) = \left(\frac{4a}{\lambda_H} + 2\mu\right)\frac{1-x}{1+x},\\
&P^u(x)S^s(x) + P^s(x)S^u(x) = -\left(2 - a - \mu\frac{1-x}{1+x}\right),\\
&P^u(x)P^s(x) = -1.
\end{aligned}
$$

where

$$
\tilde{S}^s(x) = \sum_{n=0}^{\infty} \tilde{s}_n^s x^n, \qquad \tilde{P}^s(x) = \sum_{n=0}^{\infty} \tilde{p}_n^s x^n,
$$

$$
\tilde{S}^u(x) = \sum_{n=0}^{\infty} \tilde{s}_n^u x^n, \qquad \tilde{P}^u(x) = \sum_{n=0}^{\infty} \tilde{p}_n^u x^n.
$$

Discrete transparent boundary conditions

Let $x = 1/z$

Relation between coefficients and roots

$$\tilde{S}^s(x) + \tilde{S}^u(x) = (2-a)(1+x) + \mu(1-x),$$
$$(1+x)\tilde{P}^u(x) + (1+x)\tilde{P}^s(x) + \tilde{S}^u(x)\tilde{S}^s(x) = \left(\frac{4a}{\lambda_H} + 2\mu\right)(1-x^2),$$
$$\tilde{P}^u(x)\tilde{S}^s(x) + \tilde{P}^s(x)\tilde{S}^u(x) = -\left((2-a)(1+x)^2 - \mu(1-x^2)\right),$$
$$\tilde{P}^u(x)\tilde{P}^s(x) = -(1+x)^2.$$

where

$$\tilde{S}^s(x) = \sum_{n=0}^{\infty} \tilde{s}_n^s x^n, \qquad \tilde{P}^s(x) = \sum_{n=0}^{\infty} \tilde{p}_n^s x^n,$$
$$\tilde{S}^u(x) = \sum_{n=0}^{\infty} \tilde{s}_n^u x^n, \qquad \tilde{P}^u(x) = \sum_{n=0}^{\infty} \tilde{p}_n^u x^n.$$

Discrete transparent boundary conditions

Computation of the coefficients

- Non linear system to solve for $(\tilde{s}_0^s, \tilde{p}_0^s, \tilde{s}_0^u, \tilde{p}_0^u)$,
- Linear 4×4 system to solve for $(\tilde{s}_n^s, \tilde{p}_n^s, \tilde{s}_n^u, \tilde{p}_n^u)$, $n \geq 1$.
- System invertible thanks to the separation of the roots at $x = 0$.
- Coefficients have the same behaviour as in the BBM or Schrödinger case ($n^{-3/2}$).

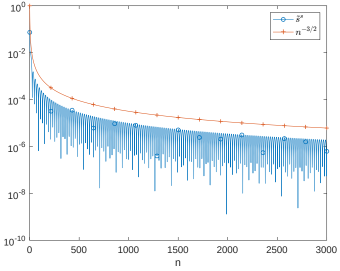

Figure: Coefficients $\tilde{s}_n^s$ with $\delta x = 2^{-18}$, $\delta t = 10^{-4}$, $\alpha = \delta = 1$ and $c = 2$

Discrete transparent boundary conditions

Computation of the coefficients

- As $\delta x \to 0$, the roots are no longer separated,
- The determinant of the system goes to zero,
- Numerical error increases
- Only for spatial steps δx smaller than in previous papers.

$\Rightarrow$ Asymptotic expansion of the coefficient as $\delta x \to 0$

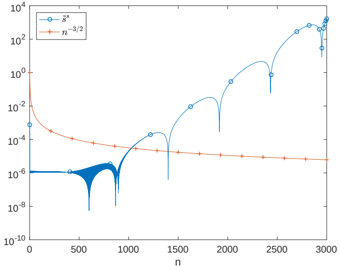

Figure: Coefficients $\tilde{s}^s_n$ with $\delta x = 2^{-18}$, $\delta t = 10^{-2}$, $\alpha = \delta = 1$ and $c = 2$

Consistency of the discrete TBC

Proposition

Let u be a smooth solution of the (KdV-BBM) system. For all $x \in [-2\delta x, 1+2\delta x]$, we define the $\mathcal{Z}$-transform of $(u(n\delta t, x))_{n\in\mathbb{N}}$ by

$$\hat{u}(z,x) = \sum_{n=0}^{\infty} \frac{u(n\delta t, x)}{z^n}.$$

Then, for all $K \subset \mathbb{C}^+$, $s \in K$, one has for the left boundary conditions:

$$\begin{aligned}
\hat{u}(e^{s\delta t}, \delta x) - S^u(e^{s\delta t})\hat{u}(e^{s\delta t}, 0) & \\
+P^u(e^{s\delta t})\hat{u}(e^{s\delta t}, -\delta x) &= \delta x^2\, O(\delta t + \delta x), \\
\hat{u}(e^{s\delta t}, 0) - S^u(e^{s\delta t})\hat{u}(e^{s\delta t}, -\delta x) & \\
+P^u(e^{s\delta t})\hat{u}(e^{s\delta t}, -2\delta x) &= \delta x^2\, O(\delta t + \delta x),
\end{aligned}$$

Consistency of the discrete TBC

Proposition

Let u be a smooth solution of the (KdV-BBM) system. For all $x \in [-2\delta x, 1+2\delta x]$, we define the $\mathcal{Z}$-transform of $(u(n\delta t, x))_{n\in\mathbb{N}}$ by

$$\hat{u}(z,x) = \sum_{n=0}^{\infty} \frac{u(n\delta t, x)}{z^n}.$$

Then, for all $K \subset \mathbb{C}^+$, $s \in K$, one has for the right boundary conditions

$$\begin{aligned}
\hat{u}(e^{s\delta t}, 1+2\delta x) - S^s(e^{s\delta t})\hat{u}(e^{s\delta t}, 1+\delta x) & \\
+P^s(e^{s\delta t})\hat{u}(e^{s\delta t}, 1) &= \delta x\, O(\delta t + \delta x), \\
\hat{u}(e^{s\delta t}, 1+\delta x) - S^s(e^{s\delta t})\hat{u}(e^{s\delta t}, 1) & \\
+P^s(e^{s\delta t})\hat{u}(e^{s\delta t}, 1-\delta x) &= \delta x\, O(\delta t + \delta x).
\end{aligned}$$

Stability of the discrete TBC

Proposition

Let u_j^n with $j \in [-1,\ J+1]$ and $n \in \mathbb{N}$ numerical solution of with the previous discrete transparent boundary conditions. Denote $\mathcal{E}_n$

$$\mathcal{E}_n = \sum_{j=1}^{J} \frac{(u_j^n)^2}{2} + \alpha \sum_{j=0}^{J} \frac{(u_{j+1}^n - u_j^n)^2}{2\delta x^2}. \tag{1}$$

There exists two hermitian matrices $\mathcal{A}^s(e^{i\theta})$ and $\mathcal{A}^u(e^{i\theta})$ such that

$$\forall N \in \mathbb{N}, \qquad \mathcal{E}_N - \mathcal{E}_0 = -\mathcal{R}_\ell - \mathcal{R}_r$$

Stability of the discrete TBC

Proposition

with

$$\mathcal{R}_r = \frac{\lambda_D}{8\pi} \int_{-\pi}^{\pi} \langle \begin{pmatrix} \widehat{u_{J-1}}(e^{i\theta}) \\ \widehat{u_J}(e^{i\theta}) \end{pmatrix} ; \mathcal{A}^s(e^{i\theta}) \begin{pmatrix} \widehat{u_{J-1}}(e^{i\theta}) \\ \widehat{u_J}(e^{i\theta}) \end{pmatrix} \rangle \, d\theta,$$

$$\mathcal{R}_\ell = \frac{\lambda_D}{8\pi} \int_{-\pi}^{\pi} \langle \begin{pmatrix} \widehat{u_{-1}}(e^{i\theta}) \\ \widehat{u_0}(e^{i\theta}) \end{pmatrix} ; \mathcal{A}^u(e^{i\theta}) \begin{pmatrix} \widehat{u_{-1}}(e^{i\theta}) \\ \widehat{u_0}(e^{i\theta}) \end{pmatrix} \rangle \, d\theta.$$

Stability of the discrete TBC

Proposition

Assume that for all $\theta \in [-\pi, \pi]$ the Hermitian matrices $\mathcal{A}^s(e^{i\theta})$ and $\mathcal{A}^u(e^{i\theta})$ are positive semi-definite. Then the transparent boundary conditions are dissipative:

$$\forall N \in \mathbb{N}, \qquad \mathcal{E}_N - \mathcal{E}_0 = -\mathcal{R}_\ell - \mathcal{R}_r \leq 0$$

with

$$\mathcal{R}_r \geq 0, \quad \mathcal{R}_\ell \geq 0.$$

This assumption are numerically satisfied

Comparison to the solution for IKdV, $\alpha + \varepsilon = 2.10^{-3}$

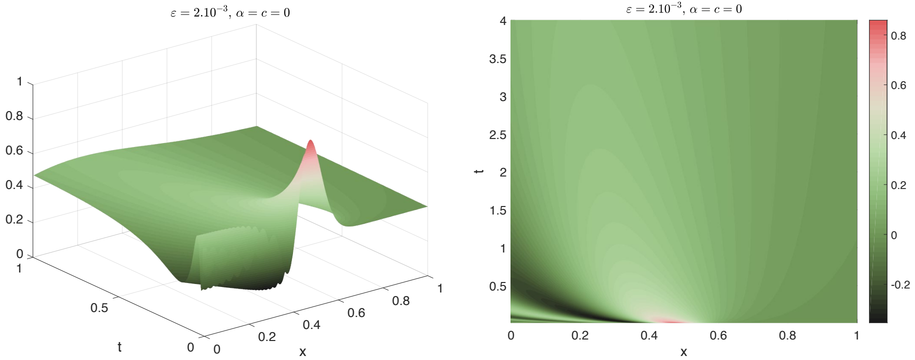

Figure: Evolution of the reference solution for $(\alpha = c = 0, \varepsilon = 2.10^{-3})$ and $u_0 = u_{0,G}$

Comparison to the solution for IKdV, $\alpha + \varepsilon = 2.10^{-3}$

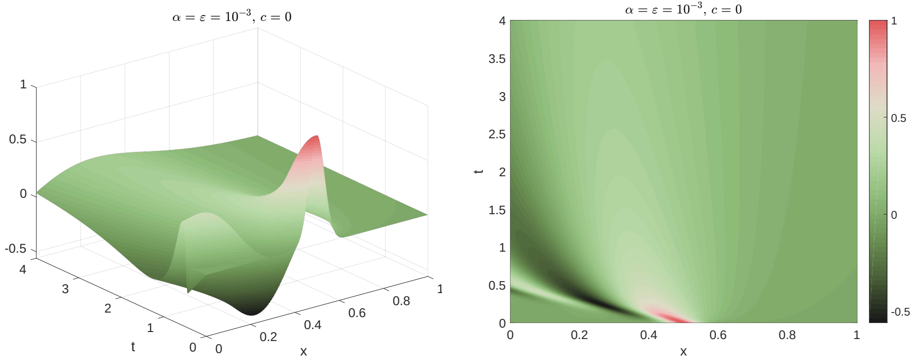

Figure: Evolution of the reference solution for $(c = 0, \alpha = \varepsilon = 10^{-3})$ and $u_0 = u_{0,G}$

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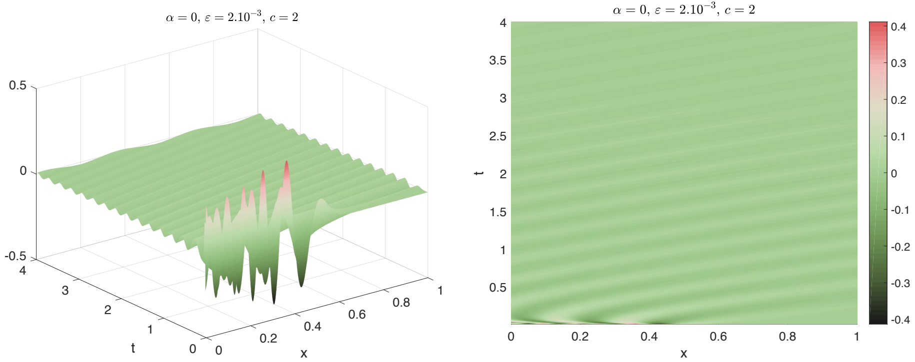


Figure: Evolution of the reference solution for $(c = 2, \alpha = 0, \varepsilon = 2.10^{-3})$ and $u_0 = u_{0,WP}$

Comparison to the solution for IKdV, $\alpha + \varepsilon = 2.10^{-3}$

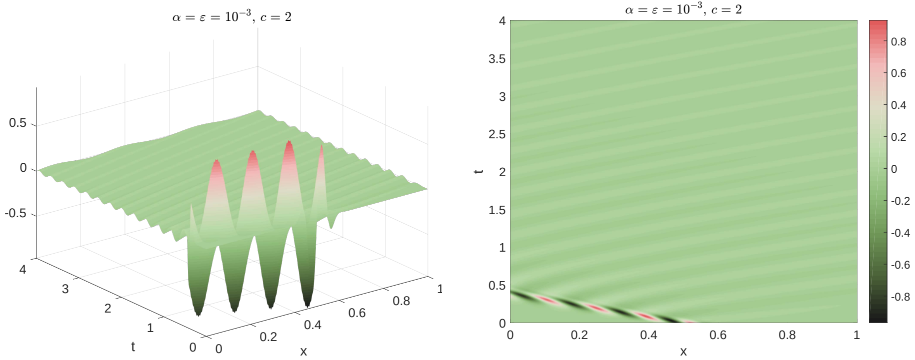
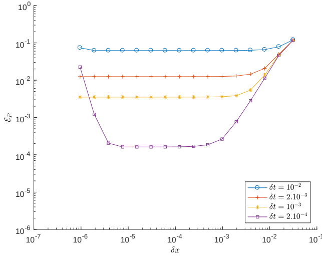
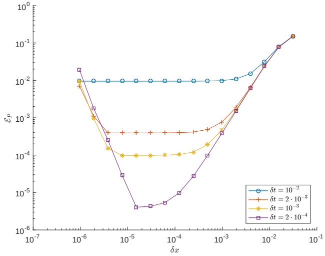


Figure: Evolution of the reference solution for $(c = 2, \alpha = \varepsilon = 10^{-3})$ and $u_0 = u_{0,WP}$

Behaviour of the relative energy with respect to δx and δt



$(\alpha = c = 0, \varepsilon = 2.10^{-3}),\ u_0 = u_{0,G}$

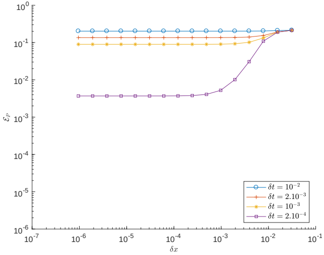


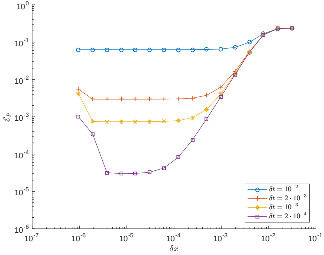
$(c = 0, \alpha = \varepsilon = 10^{-3}),\ u_0 = u_{0,G}$

Figure: Evolution of $\mathcal{E}_P$ with respect to δx for various δt.

- As $\delta x < 5.10^{-5}$, bad behaviour of $\mathcal{E}_P$
- Inversion of a matrix whose determinant is of order $\mathcal{O}\left(\frac{c\delta x^2}{\varepsilon} + \frac{\delta x^3}{\varepsilon \delta t}\right)$

Behaviour of the relative energy with respect to δx and δt





$(c = 2, \alpha = 0, \varepsilon = 2.10^{-3}), u_0 = u_{0,WP}$ $\quad$ $(c = 2, \alpha = \varepsilon = 10^{-3}), u_0 = u_{0,WP}$

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Approximate discrete transparent boundary conditions

Recall that the problem of inverting the $\mathcal{Z}$-transform in the transparent boundary conditions amounts to expand into Laurent series the functions $s^s(z), s^u(z), p^s(z), p^u(z)$ defined by the relation

$$
\begin{aligned}
P(r) &= r^4 - 2r^3 + \frac{4\delta x^3}{\varepsilon \delta t} p(z) r^2 + 2r - 1 \\
&= \left(r^2 - s^s(z) r + p^s(z)\right)\left(r^2 - s^u(z) r + p^u(z)\right).
\end{aligned}
$$

The roots of $r^2 - s^s r + p^s$ belongs to $\{r \in \mathbb{C},\ |r| < 1\}$ whereas the ones of $r^2 - s^u r + p^u$ belongs to $\{r \in \mathbb{C},\ |r| > 1\}$.

Approximate discrete transparent boundary conditions

Let us calculate (s^s, p^s, s^u, p^u). These functions satisfy

$$\begin{cases} s^s + s^u & = & 2, \\ s^s s^u + p^s + p^u & = & \frac{4\delta x^3}{\varepsilon \delta t} p(z), \\ s^s p^u + s^u p^s & = & -2, \\ p^s p^u & = & -1. \end{cases}$$

We look for an asymptotic expansion of these quantities as $\delta x \to 0$ in the form:

$$s^s = \sum_{k \geq 0} s_k \delta x^k, \quad p^s = \sum_{k \geq 0} p_k \delta x^k, \quad s^u = \sum_{k \geq 0} t_k \delta x^k, \quad p^u = \sum_{k \geq 0} q_k \delta x^k.$$

Approximate discrete transparent boundary conditions

By inserting this expansion into the equation and identifying $O(\delta x^n)$ terms with $(n \in \mathbb{N})$, we obtain a non linear system at 0^{th} order:

$$\left\{\begin{array}{lcl} s_0 + t_0 & = & 2, \\ s_0 t_0 + p_0 + q_0 & = & 0, \\ s_0 q_0 + t_0 p_0 & = & -2, \\ p_0 q_0 & = & -1. \end{array}\right.$$

The solution writes $(s_0, p_0, t_0, q_0) = (0, -1, 2, 1)$.

Approximate discrete transparent boundary conditions

Next, we identify $O(\delta x^n)$ terms with $n \geq 1$. One finds the family of linear systems

$$A \begin{pmatrix} s_n \\ p_n \\ t_n \\ q_n \end{pmatrix} = F_n \quad \text{where } A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ t_0 & 1 & s_0 & 1 \\ q_0 & t_0 & p_0 & s_0 \\ 0 & q_0 & 0 & p_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix},$$

where 0 is a simple eigenvalue associated to $v = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$.

Approximate discrete transparent boundary conditions

If the compatibility condition

$$\det \left(F_n, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right) = 0$$

is fulfilled, then one can compute $U_n = (s_n, p_n, t_n, q_n)^T$.

Approximate discrete transparent boundary conditions

Let λ_1 the root of $\lambda_1^3 + \frac{2}{\varepsilon\delta t}p(z) = 0$ whose real part is negative. We get:

$$s^s = \lambda_1\delta x + \frac{\lambda_1^2}{2}\delta x^2 + \frac{p}{3\varepsilon\delta t}\delta x^3 + O(\delta x^4),$$

$$s^u = 2 - \lambda_1\delta x - \frac{\lambda_1^2}{2}\delta x^2 - \frac{p}{3\varepsilon\delta t}\delta x^3 + O(\delta x^4),$$

$$p^s = -1 - \lambda_1\delta x - \frac{\lambda_1^2}{2}\delta x^2 + \frac{2p}{3\varepsilon\delta t}\delta x^3 + O(\delta x^4),$$

$$p^u = 1 - \lambda_1\delta x + \frac{\lambda_1^2}{2}\delta x^2 + \frac{2p}{3\varepsilon\delta t}\delta x^3 + O(\delta x^4).$$

Approximate discrete transparent boundary conditions

We now need to invert the $\mathcal{Z}$ transform of $z \mapsto \lambda_1(s(z)) = -\left(\frac{2}{\varepsilon\delta t}\right)^{1/3} p(z)^{1/3}$. Note that

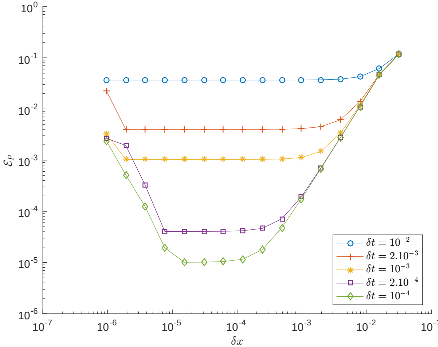
$$p(z)^{k/3} = \frac{(1 - z^{-1})^{k/3}}{(1 + z^{-1})^{k/3}}, \quad \forall |z| > 1, \quad \forall k \in \mathbb{Z}.$$

As a consequence, $p(z)^{k/3}$ can be expanded into Laurent series explicitly: indeed, $(1 - z^{-1})^{\gamma}$ and $(1 + z^{-1})^{\gamma}$ expand as

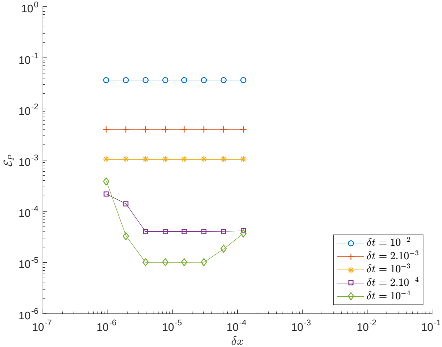
$$(1 - z^{-1})^{\gamma} = \sum_{p=0}^{\infty} \frac{\alpha_p^{(\gamma)}}{z^p}, \quad \alpha_{p+1}^{(\gamma)} = -\frac{\gamma - (p-1)}{p}\alpha_p^{(\gamma)}, \quad \alpha_0 = 1,$$

$$(1 + z^{-1})^{\gamma} = \sum_{p=0}^{\infty} \frac{\beta_p^{(\gamma)}}{z^p}, \quad \beta_{p+1}^{(k)} = \frac{\gamma - (p-1)}{p}\beta_p^{(\gamma)}, \quad \beta_0 = 1.$$

Numerical results- ($\alpha = c = 0, \varepsilon = 10^{-3}$), $u_0 = u_{0,G}$



standard coefficients

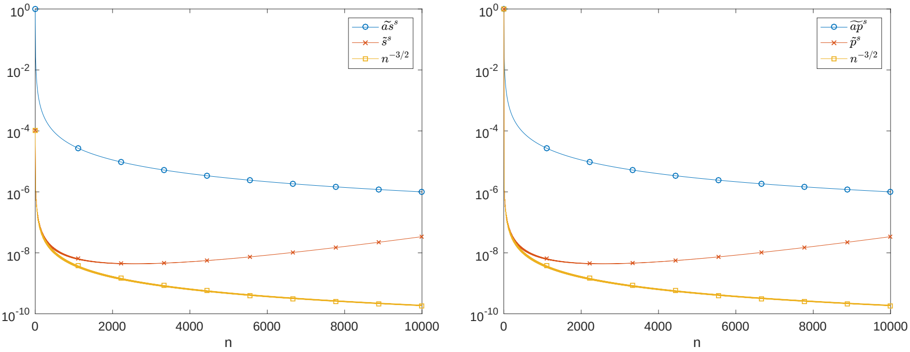


asymptotic coefficients

Figure: Evolution of $\mathcal{E}_P$ with respect to δx for various δt.

The bad behaviour of $\mathcal{E}_P$ is clearly limited when $\delta x, \delta x^3/\delta t$ are very small.

Numerical results - Long time simulations

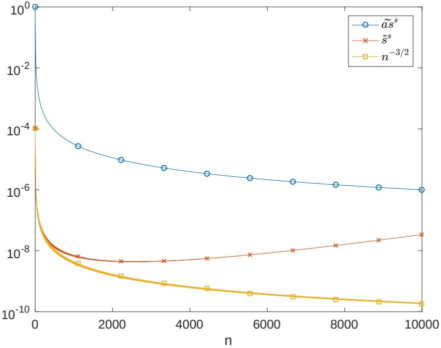


coefficients $\tilde{p}^s$ and $\widetilde{ap}^s$ coefficients $\tilde{s}^s$ and $\widetilde{as}^s$

Figure: Evolution of the convolution coefficients.

- Asymptotic coefficients useful for long time simulations
- Standard coefficient do not have the good decay $(n^{-3/2})$

Numerical results



coefficients $\tilde{p}^s$ and $\widetilde{ap}^s$

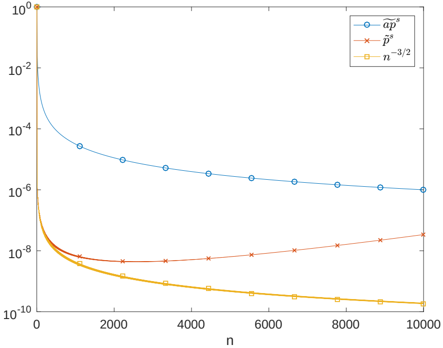


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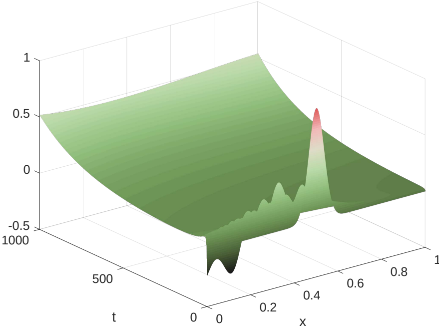
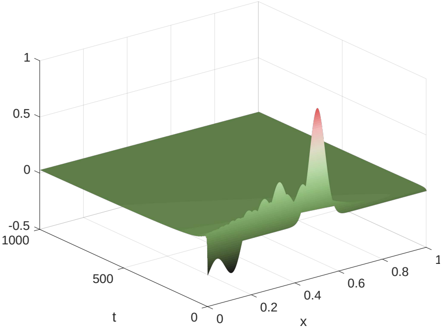
Figure: Evolution of the convolution coefficients.

Numerical results - Long time simulations



standard coefficients



asymptotic coefficients

Figure: Evolution of the solution with standard and asymptotic convolution coefficients.

Numerical results - Long time simulations

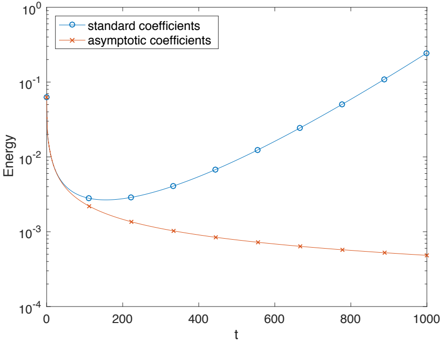


Figure: Evolution of the discrete energy $\mathcal{E}_n$ of the solution with standard and asymptotic convolution coefficients.

Conclusion

- Continuous and discrete transparent boundary conditions for the linear mixed KdV-BBM equation
- Second order in time and space scheme
- Preserves spatial mean ad energy
- Stability for the continuous transparent boundary conditions
- Sufficient condition in the discrete case for the stability
- Consistence between the discrete and continuous transparent boundary conditions

Conclusion

- New strategy to compute the inverse $\mathcal{Z}$-transform, based on an asymptotic expansion as $x = 1/z \to 0$
- Method efficient and stable except for small δx
- Alternative strategy based on an asymptotic expansion as $\delta x \to 0$.
- Coefficients have good behaviour for long time simulations

Perspectives

- Non linear equations
  - Equations with variable coefficients
  - Fixed point method
- Design of discret transparent boundary conditions for more general models of water waves (KP, Zakharov-Kuznetsov, Serre-Green-Naghdi)

Thanks for your attention